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# Field equations in teleparallel space–time: Einstein’s *Fernparallelismus* approach toward unified field theory

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## Abstract

A historical account of Einstein’s *Fernparallelismus* approach toward a unified field theory of gravitation and electromagnetism is given. In this theory, a space–time characterized by a curvature-free connection in conjunction with a metric tensor field, both defined in terms of a dynamical tetrad field, is investigated. The approach was pursued by Einstein in a number of publications that appeared in the period from summer 1928 until spring 1931. In the historical analysis special attention is given to the question of how Einstein tried to find field equations for the tetrads. We claim that it was the failure to find and justify a uniquely determined set of acceptable field equations that eventually led to Einstein’s abandoning this approach. We comment on some historical and systematic similarities between the *Fernparallelismus* episode and the *Entwurf* theory, i.e., the precursor theory of general relativity pursued by Einstein in the years 1912–1915.

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## Zusammenfassung

Es wird eine historische Darstellung von Einsteins *Fernparallelismus* Ansatz zu einer einheitlichen Feldtheorie gegeben. Bei diesem Ansatz ist die Raumzeit durch einen krümmungsfreien Zusammenhang in Verbindung mit einem metrischem Tensorfeld bestimmt, wobei beide durch ein dynamisches Tetradenfeld definiert sind. Der Ansatz wurde von Einstein in einer Reihe von Publikationen aus der Zeit von Sommer 1928 bis Frühjahr 1931 verfolgt. In der historischen Analyse wird der Frage besondere Beachtung geschenkt, auf welche Weise Einstein versuchte, Feldgleichungen für die Tetraden aufzustellen. Es wird die These vertreten, dass Einstein den Ansatz schliesslich aufgab, weil es ihm nicht gelang, eindeutig bestimmte Feldgleichungen zu finden und zu begründen. Schliesslich werden einige historische und systematische Ähnlichkeiten zwischen der *Fernparallelismus*-Episode und der sogenannten Entwurf-Theorie, d.i. der Vorläufertheorie der allgemeinen Relativitätstheorie diskutiert.

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## 1. Introduction

Einstein's attempt to base a unified theory of the gravitational and electromagnetic fields on the mathematical structure of distant parallelism, also referred to as absolute parallelism or teleparallelism,<sup>1</sup> was an episode that lasted for three years, from summer 1928 until spring 1931. The crucial new concept, for Einstein, that initiated the approach was the introduction of the tetrad field, i.e., a field of orthonormal bases of the tangent spaces at each point of the four-dimensional manifold. The tetrad field was introduced to allow the distant comparison of the direction of tangent vectors at different points of the manifold; hence the name distant parallelism. From the point of view of a unified theory, the specification of the four tetrad vectors at each point involves the specification of 16 components instead of only 10 for the symmetric metric tensor. The idea then was to exploit the additional degrees of freedom to accommodate the electromagnetic field. Mathematically, the tetrad field easily allows the conceptualization of more general linear affine connections, in particular, nonsymmetric connections of vanishing curvature but nonvanishing torsion. Since, however, Einstein wanted to combine a curvature-free connection with a nontrivial metric the resulting structure actually involves two different connections and a certain ambiguity was inherent in their interpretation.

The published record of the distant parallelism episode comprises eight papers in the *Sitzungsberichte* of the Prussian Academy, at the time Einstein's major forum for publication of scientific results. A review paper on the theory appeared in the *Mathematische Annalen*, a leading mathematics journal, together with a historical essay on the subject matter by Elie Cartan. The theory of distant parallelism was also touched upon in popular articles by Einstein for the New York and London *Times*. Accounts that place his new attempt in a larger tradition of field-theoretic attempts in the history of physics are made in a contribution to a *Festschrift* for Aurel Stodola and in three popular papers on the *Raum-, Feld-, und Äther-Problem* in physics. Two of the papers, including the very last one, were coauthored with Walter Mayer, Einstein's mathematical collaborator. The episode is also reflected in Einstein's contemporary correspondence, notably with Herman Müntz, Roland Weitzenböck, Cornelius Lanczos, Elie Cartan, and Walther Mayer. For most of the published papers manuscript versions are extant in the Einstein Archives and there are also a number of unidentified and undated research calculations that are related to the distant parallelism approach.<sup>2</sup>

As far as Einstein was involved in it, the *Fernparallelismus* approach has a distinct beginning, a period of intense investigation, and a somewhat less distinct but definite end. The mathematical structure in question had been developed before by others, notably by Elie Cartan and Roland Weitzenböck, in other,

<sup>1</sup> In this paper, these terms will be used interchangeably.

<sup>2</sup> See especially the documents with archive numbers 62-001ff. A thorough investigation of the unpublished correspondence and scientific manuscripts remains to be done; see [Sauer, 2004]. In this paper, I will largely rely on the published record of the episode.

purely mathematical contexts.<sup>3</sup> Einstein's pursuit of the approach also triggered a more general discussion that involved quite a few other contemporary physicists and mathematicians,<sup>4</sup> and it continued to be investigated by others even when Einstein no longer took active part in these discussions. Even today, teleparallelism is occasionally discussed, e.g., as a rather special case in a more general conceptual framework of a metric–affine gauge theory of gravity.<sup>5</sup> And although not considered in contemporary mainstream physics a viable approach for any realistic physics, it is nevertheless still discussed as one of several options for going beyond standard Riemannian geometry by authors who investigate generalized geometrical frameworks in the context of a search for a quantum theory of gravitation.<sup>6</sup> Indeed, it is in the background of contemporary attempts at grand unification schemes that Einstein's later work on his unified field theory program—which under the spell of Abraham Pais's verdict that “these attempts have led nowhere” [Pais, 1982, 336] was long neglected by historians of physics—warrants, I believe, some serious historical investigation.

In this paper, a historical account of the *Fernparallelismus* approach is given as an episode in Einstein's intellectual life.<sup>7</sup> The account will largely be organized chronologically and give the relevant biographical data, as it were, of the life-cycle of this approach, as far as Einstein is concerned. The infancy of the approach, Section 2, is given by Einstein's first two notes on distant parallelism, which lay out the mathematical structure and give a first derivation of a set of field equations. I will give a brief systematic characterization of the approach in its historical context in Section 2.1<sup>8</sup> and discuss Einstein's first notes in Section 2.2. In its early childhood, Einstein entered into interaction with mathematicians and learnt about earlier pertinent developments in mathematics, Section 3. I will briefly comment on his correspondence with Herman Müntz, Section 3.1, and Roland Weitzenböck, Section 3.2, as well as on his collaboration with Jakob Grommer and Cornelius Lanczos, Section 3.3. The period of adolescence is primarily concerned with the problem of finding field equations, Section 4. I first discuss the publication context of Einstein's next papers in Section 4.1 and then focus on his attempts to find and justify a set of field equations in Section 4.2. Einstein here wavered between a variational approach and an approach where field equations were determined utilizing algebraic identities for an overdetermination of the equations, Section 4.3. The mature stage is reached when Einstein settled on a set of field equations and wrote an overview of the theory published in the *Mathematische Annalen*, Section 5. I will first give an account of the publication history of this paper which is intimately linked with Einstein's correspondence with Elie Cartan, Section 5.1, and then discuss the derivation of the field equations, Section 5.2. In its old age, Section 6, Einstein improved on the compatibility proof, Section 6.1, promoted the theory in public and defended it against criticism, Section 6.2, and explored its consequences, Section 6.3. The last sign

<sup>3</sup> I emphasize that I will refrain from any claims regarding questions of mathematical priority with respect to the concepts of differential geometry relevant to the episode.

<sup>4</sup> See [Goldstein and Ritter, 2003, 120ff.].

<sup>5</sup> See, e.g., [Gronwald, 1997, esp. 295ff]; see also [Hehl et al., 1980], which lists a number of references that have taken up the distant parallelism approach over time on p. 341.

<sup>6</sup> See, e.g., [Borzeszkowski, 2002].

<sup>7</sup> For earlier historical discussions of the *Fernparallelismus* approach, see [Treder, 1971, 60–67; Pais, 1982, 344–347; Biezunski, 1989; Bergia, 1993, 292–294; Vizgin, 1994, 234–257; Goldstein and Ritter, 2003, 120–133; van Dongen, 2002a, 57–58; Goenner, 2002, Section 4.3.3]. See also [Goenner, 2004, Section 6.4], which was published after submission of this paper.

<sup>8</sup> A concise characterization of teleparallelism as a geometrical structure in modern terms is given in Appendix B.

of life is a paper that systematically investigated compatible field equations in a teleparallel space–time, Section 6.4.

The life-cycle of the distant parallelism approach bears a number of striking similarities to the life-cycle of the *Entwurf* theory; i.e., the precursor theory of general relativity advanced and pursued by Einstein between 1912 and 1915 and presented first in an *Outline* (“*Entwurf*”) of a *Generalized Theory of Relativity and a Theory of Gravitation* in 1913 [Einstein and Grossmann, 1913]. Some of these similarities between the history of the *Fernparallelismus* approach and the *Entwurf* theory will be pointed out along the way. They are, I believe, no coincidence. I will offer some reflections on the systematic reason for this similarity in the concluding remarks in Section 7.

## 2. Einstein’s distant parallelism as a mathematical structure

Before entering into the discussion of the historical material, the mathematical and conceptual framework of Einstein’s distant parallelism shall be characterized briefly from a historical and systematic viewpoint.<sup>9</sup> I will then discuss Einstein’s first two notes on the subject.

### 2.1. Distant parallelism in the historical context of Einstein’s unified field theory program

Einstein’s breakthrough to general covariance and the closure of his long search for a relativistic theory of gravitation with the publication of the Einstein equations in November 1915 were a victory of the concept of a metric tensor and its conceptual role for Einstein’s new interpretation of gravity.<sup>10</sup> It was not necessarily a victory of a geometrization of physics nor of a search for a unification of physics.

For one, the crucial concepts of the initial formulation of general relativity like the Riemann and Ricci tensors were largely interpreted, at least by Einstein, from a purely algebraic and invariant-theoretic point of view. It was only through the efforts of mathematicians such as David Hilbert, Felix Klein, Hermann Weyl, Tullio Levi-Civita, Gustav Herglotz, Hermann Vermeil, and others, that the geometric implications of those concepts were gradually recognized.<sup>11</sup> A significant step in this direction had been made by focussing on the concept of an affine connection, mainly by Levi-Civita and Weyl, and by interpreting the curvature of a generalized Riemannian metric space in terms of parallel transport of vectors. Indeed, although closely related, the concepts of a metric tensor field and of an affine connection are conceptually quite independent. The metric tensor field allows one to determine a vector’s length at each point of a manifold and consequently permits the definition of concepts like an angle between vectors and of physical properties like spatial distances and time intervals. But in and of itself the metric alone does not permit the comparison of two vectors in different tangent spaces other than with respect to their length. Instead, it is the concept of an affine connection that is needed to identify vectors in nearby tangent spaces or, by means of integration along a trajectory of parallel transport, in tangent spaces

<sup>9</sup> For a characterization of the geometric structure of distant parallelism in modern coordinate-independent terms, see [Appendix B](#).

<sup>10</sup> For an account of Einstein’s discovery of general relativity from the point of view of the history of mathematics, see [Sauer, 2005a]; see also [Renn et al., forthcoming] and further references cited in these works.

<sup>11</sup> For historical accounts of the immediate reception of the theory of general relativity by mathematicians; see [Reich, 1992; Rowe, 1999].

at arbitrary points of the manifold. The connection and its associated concepts like parallel transport, covariant differentiation, and curvature, therefore turn into a fundamental concept for the elaboration of generalized geometries. But the connection, in turn, does not carry information about the metric structure. However, the requirement of compatibility between metric and affine structures poses constraints on both concepts. In any case, it was only in the course of elaborating the concepts and mathematical structure of Einstein's general theory of relativity of 1915 that the affine connection emerged as a crucial concept in its own right.<sup>12</sup>

Nor was Einstein's breakthrough to general relativity in November 1915, as the competition with Hilbert in those final weeks shows, anything like a victory of a unification of the gravitational and electromagnetic forces.<sup>13</sup> But the coming into being of the general theory of relativity created a general framework for a research program of geometrized unification of those two forces as well as for more ambitious programs that also aimed at overcoming the duality of fields and matter.<sup>14</sup> However, the Einstein gravitational field equations or equivalently the Hilbert action in a variational formulation are virtually uniquely determined by the physical requirements of a relativistic theory of gravitation. Attempts at a unification therefore had to transcend the semi-Riemannian framework of a four-dimensional space–time in order to entertain any hope of achieving their goals.

The program of unification of the gravitational and electromagnetic fields after the advent of general relativity therefore arose as an intricate interplay between aspects of a mathematical representation and its physical interpretation. On one level, the mathematical representation had to provide the dynamical variables in terms of which the gravitational and electromagnetic fields would be expressed, and equations for those variables had to be found that would reduce in some way to the known gravitational and electromagnetic field equations, i.e., would reproduce Einstein's and Maxwell's equations, at least in some appropriate limit. The embedding of the representation in a mathematical context, on the other hand, also had to ensure that the unification was not only formal but arose in some sense organically out of an embracing mathematical framework. Precisely, what the criteria for a successful unification of the known physical forces of gravitation and electromagnetism had to be was not a canonically agreed upon set of aims but depended, to some extent, on the respective authors and on the respective approach.

Weyl's unified field theory figures most prominently as the first attempt at a geometrized unification of gravitation and electromagnetism that seemed to promise some success.<sup>15</sup> Starting from a philosophically motivated concept of a purely infinitesimal geometry, Weyl questioned the metric-compatible Levi-Civita connection as a fundamental concept since it allowed the distant comparison of lengths. Thus, parallel transport around a closed loop using the Levi-Civita connection would always preserve the vector's length, whereas the deficit angle upon return to the origin would depend on the manifold's curvature and the specific path of the loop. Substituting the metric field by the class of all conformally equivalent metrics, Weyl therefore introduced a connection that would not carry any information about the length of a vector on parallel transport. Instead the latter task was assigned to an extra connection, a so-called length connection that would, in turn, not carry any information about the direction of a vector on parallel

<sup>12</sup> See [Reich, 1992].

<sup>13</sup> For a discussion of Einstein's and Hilbert's respective contributions in November 1915, see [Sauer, 1999, 2005b] and further references cited therein.

<sup>14</sup> For historical accounts of Einstein's unified field theory program, see [Bergia, 1993; Vizgin, 1994; van Dongen, 2002a; Goenner, 2004; Sauer, forthcoming].

<sup>15</sup> For a discussion of Weyl's work, see [Vizgin, 1994, Chapter 3; Scholz, 2001].

transport but that would only fix or gauge the conformal factor. As it turned out, Weyl was able to identify the length connection with the electromagnetic potential, but other problems of physical interpretation were pointed out quickly by Einstein.

From the point of view of the emergence of Einstein's program of a unified field theory, two other approaches to transcend the framework of semi-Riemannian geometry were significant. One such approach was Theodor Kaluza's idea to increase the number of dimensions of space–time to accommodate the electromagnetic field by means of the additional degrees of freedom in a five-dimensional space–time.<sup>16</sup> The other option is associated with the name of Arthur Stanley Eddington and is given by the idea of taking the coefficients of the affine connection as the fundamental dynamic variables of the theory rather than the components of the metric.<sup>17</sup> Einstein picked up and elaborated on both approaches in a number of publications of the early 1920s.

The significance of the affine connection as an independent mathematical concept is also at the core of Einstein's first own original attempt at a unified field theory which he published in 1925.<sup>18</sup> Here he takes both the metric and the affine connection as independent dynamical variables and investigates the implied mathematical structure with respect to his aims of unification. This so-called affine approach of 1925 turned out to be shortlived at the time, but it would be taken up again as the last approach that he followed during the last ten years of his life.

After publishing two papers on Kaluza's five-dimensional approach in 1927, Einstein made distant parallelism approach his next major attempt at a geometrized unified field theory. As far as its mathematical foundation is concerned, Einstein, as we will see, discovered a way to generalize semi-Riemannian geometry that had already been investigated by several mathematicians in purely mathematical contexts (see Sections 3 and 5.1). For Einstein it was the discovery that the concept of a tetrad field opened up new vistas for a generalized framework. In his own conceptual justification of the approach he made reference to the geometric implications.

He pointed out that in Weyl's *Infinitesimalgeometrie* neither lengths nor directions of vectors would be comparable without explicit specification of an affine and of a length connection respectively. Riemannian geometry, on the other hand, allowed the comparison of lengths over finite distances by virtue of the metric tensor field, but not of directions. In contrast to Weyl's geometry and Riemannian geometry, his own theory of distant parallelism, Einstein said, allowed distant comparison of both direction and lengths. The comparison is problematic because from a modern point of view, it would seem more natural to parallelize the *Fernparallelismus* to Weyl's theory as two different ways of generalizing the affine connection that determines the geometrical properties.<sup>19</sup> Indeed, this way of characterizing distant parallelism immediately raises the question of the difference between distant parallelism and classic Euclidean geometry.

As was pointed out quickly by Hans Reichenbach [Reichenbach, 1929], a better way of characterizing the geometric structure of distant parallelism is by saying that it is a space endowed with a general, i.e., nonsymmetric real affine connection with vanishing Riemann–Cartan curvature. Technically, the antisymmetric part of the connection defines a tensor that is called the torsion tensor, which vanishes in Riemannian geometry, where only real and symmetric connections are considered. Indeed, the difference

<sup>16</sup> For historical accounts, see [van Dongen, 2002b; Goenner and Wünsch, 2003].

<sup>17</sup> For historical accounts, see [Goenner, 2004, Section 6.1; Majer and Sauer, 2005].

<sup>18</sup> See [Vizgin, 1994, pp. 204–219] and [Goenner, 2004, Section 6.1].

<sup>19</sup> See [Reichenbach, 1929; Goldstein and Ritter, 2003, p. 121] for further discussion of this point.



between teleparallel and Euclidean geometry is captured by the possibility of nonvanishing torsion in the former. Parallel transport of a vector along a closed loop in a teleparallel geometry would reproduce the original vector without change of direction and hence indicate that the space was Riemann flat, i.e., that the Riemannian curvature vanished. But parallel transport of a vector along the edges of a parallelogram would fail to bring the vector back to its original position and the distance between the initial and final points of a parallelogram provides a measure of the torsion.<sup>20</sup>

Here then was a way to generalize the semi-Riemannian framework of the conventional theory of general relativity by taking as fundamental dynamical variables the components of a tetrad field. This allowed Einstein to define a metric in the usual sense and thus make contact with the original gravito-inertial interpretation of gravitation. It also allowed him to define geometric concepts such as the torsion field that might be associated in some way with the electromagnetic field. But from the outset, both the geometric properties and the inherent constraints for an interpretation in terms of physical concepts were unclear.

In more modern terms, the basic ingredients of Einstein's distant parallelism are a curvature-free connection—often called a Weitzenböck connection—and the demand of global  $SO(n-1, 1)$ -symmetry.<sup>21</sup> The Weitzenböck connection defines a frame field, and the global rotation symmetry assures that the frame field can determine a metric tensor field in a meaningful way. But since a given metric field would not uniquely determine a frame field, Einstein had thus introduced a surplus structure which he hoped to be able to exploit for setting up a unified field theory of the gravitational and electromagnetic fields.

In 1912, it was the metric tensor that opened up new possibilities for exploring a generalized theory of relativity and a field theory of gravitation. Similarly new vistas had been opened by taking the concept of a (symmetric) connection as the new basic mathematical ingredient, especially in his metric-affine theory of 1925. Now it was the introduction of a tetrad field that provided new possibilities as well as new constraints.

Since two distinct connections are involved in the structure, a certain ambiguity is involved as to which connection is the physically meaningful one. But this ambiguity may not become explicit in a unified theory of the kind that Einstein was searching for, since no external matter fields are assumed for which one would have to decide which connection should determine covariant differentiation. Also it turned out that it was not easy to determine how the electromagnetic field is to be defined in terms of the frame field. Finally and most importantly, to set up a physically meaningful structure the frame field needs to be determined by some set of field equations.

## 2.2. *Einstein's first two notes*

The episode of *Fernparallelismus*, as far as Einstein is concerned, begins with two rather short notes, 5 and 4 pages, published within a week's interval in the *Sitzungsberichte* of the Prussian Academy. The first note is entitled "Riemannian geometry, maintaining the concept of distant parallelism" [Einstein, 1928a] and was presented to the Academy on June 7, 1928.

<sup>20</sup> See also the discussion below in Section 4.1.

<sup>21</sup> See Appendix B for a characterization of distant parallelism in modern, coordinate-free terms.

Einstein at the time suffered from a serious illness of the heart.<sup>22</sup> He had experienced a circulatory collapse in Switzerland in March. An enlargement of the heart was diagnosed and, back in Berlin, he was ordered strict bed rest as well as a salt-free diet and diuretics. At the end of May, he wrote to his friend Zangger: “In the tranquility of my sickness, I have laid a wonderful egg in the area of general relativity. Whether the bird that will hatch from it will be vital and long-lived only the Gods know. So far I am blessing my sickness that has endowed me with it.”<sup>23</sup> Since he was feeling too weak to attend the Academy meetings, his note was presented to the Academy by Max Planck.

The paper explains the notion of a tetrad field (“*n*-Bein”-field) and of distant parallelism (“Fernparallelismus”) for a manifold of  $n$  dimensions. The tetrad field is introduced in terms of components  $h_s^\nu$  of its vectors with respect to the naturally induced coordinate basis. Hence  $h_s^\nu$  denotes the  $\nu$ -component of the vector  $s$  with respect to the local coordinate chart. Einstein uses Greek letters to denote the coordinate indices (“Koordinaten-Indizes”) and Latin letters to denote the tetrad indices (“Bein-Indizes”). In modern literature, these indices are also referred to as holonomic respectively anholonomic. We have the relations

$$h_{a\mu}h^{av} = \delta_\mu^v, \quad (1)$$

$$h_{a\mu}h_b^\mu = \delta_{ab}, \quad (2)$$

where summation over repeated indices is always implied.<sup>24</sup>

Einstein emphasized that the tetrads define both the metric and the distant parallelism simultaneously:

By means of the introduction of the  $n$ -Bein field both the existence of a Riemann metric and the existence of the distant parallelism is expressed.<sup>25</sup>

The components of the metric tensor  $g_{\mu\nu}$  are given as

$$g_{\mu\nu} = h_\mu^a h_{a\nu}. \quad (3)$$

By virtue of (3) coordinate indices are raised and lowered using the metric  $g_{\mu\nu}$ , whereas by (2) tetrad indices are raised and lowered using  $\delta_{ab}$ .

Parallel transport is defined through the tetrads, in the sense that a vector with components  $A_a$  at one point will be parallel to a vector  $A'_a$  at another point if the components with respect to the respective

<sup>22</sup> For the following biographical information, see [Fölsing, 1997, pp. 600–607].

<sup>23</sup> “Ich habe in der Ruhe der Krankheit ein wundervolles Ei gelegt auf dem Gebiete der allgemeinen Relativität. Ob der daraus schlüpfende Vogel vital und langlebig sein wird, liegt noch im Schosse der Götter. Einstweilen segne ich die Krankheit, die mich so begnadet hat” [Einstein to Zangger, end of May 1928, Einstein Archives, The Hebrew University of Jerusalem (EA), call no. 40-069].

<sup>24</sup> Einstein’s original notation did not distinguish between tetrad vectors and the canonical dual covectors; i.e., he did not use superscripted tetrad indices. In general, I will not adhere strictly to Einstein’s original notation. In particular, I will denote partial coordinate derivatives by comma-delimited subscripts.

<sup>25</sup> “Durch die Setzung des  $n$ -Bein-Feldes wird gleichzeitig die Existenz der Riemann-Metrik und des Fernparallelismus zum Ausdruck gebracht” [Einstein, 1928a, p. 218].



tetrads are the same. The law of parallel transport is hence given by the condition

$$0 = dA_a = d(h_{a\mu}A^\mu) = h_{a\mu,\sigma}A^\mu dx^\sigma + h_{a\mu}dA^\mu. \quad (4)$$

Multiplication with  $h^{av}$  turns this into

$$dA^v = -\Delta_{\mu\sigma}^v A^\mu dx^\sigma, \quad (5)$$

where the connection

$$\Delta_{\mu\sigma}^v = h^{av}h_{a\mu,\sigma} \quad (6)$$

is introduced.<sup>26</sup> As Einstein noted, it is “rotation-invariant” and asymmetric in its lower indices. Parallel transport along a closed line reproduces the same vector; i.e., the Riemann curvature,

$$R_{\kappa\lambda\mu}^\iota = -\Delta_{\kappa\lambda,\mu}^\iota + \Delta_{\kappa\mu,\lambda}^\iota + \Delta_{\alpha\lambda}^\iota \Delta_{\kappa\mu}^\alpha - \Delta_{\alpha\mu}^\iota \Delta_{\kappa\lambda}^\alpha \equiv 0, \quad (7)$$

vanishes identically.

Einstein observed that the metric (3) gives rise to another, nonintegrable law of parallel transport, which is determined by the symmetric Levi-Civita connection,

$$\Gamma_{\mu\sigma}^v = \frac{1}{2}g^{v\alpha}(g_{\mu\alpha,\sigma} + g_{\sigma\alpha,\mu} - g_{\mu\sigma,\alpha}). \quad (8)$$

He also introduced the contorsion tensor  $\Gamma_{\alpha\beta}^v - \Delta_{\alpha\beta}^v$  and the torsion tensor

$$\Lambda_{\alpha\beta}^v = \frac{1}{2}(\Delta_{\alpha\beta}^v - \Delta_{\beta\alpha}^v) \quad (9)$$

$$= \frac{1}{2}h^{av}(h_{a\alpha,\beta} - h_{a\beta,\alpha}), \quad (10)$$

although he does not use those names for these quantities.

The possibility of obtaining field equations from a variational principle,

$$\delta \int \{\mathcal{H} d\tau\} = 0, \quad (11)$$

is briefly indicated. The variation would have to be done with respect to the 16 quantities  $h_{a\mu}$  and the Lagrangian  $\mathcal{H}$  would have to be a linear function of the two invariants  $g^{\mu\nu}\Lambda_{\mu\beta}^\alpha\Lambda_{\nu\alpha}^\beta$  and  $g_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}\Lambda_{\alpha\beta}^\mu\Lambda_{\sigma\tau}^\nu$ , multiplied by the determinant  $h = |h_{a\mu}|$ , since  $h d\tau$  is an invariant volume element.

The second note is entitled “New possibility for a unified field theory of gravitation and electricity” [Einstein, 1928b] and was presented to the Academy only a week after the first paper, on 14 June 1928.

<sup>26</sup> As pointed out already by [Reichenbach, 1929, p. 687] Einstein’s original paper contained several typographical errors in these equations; see also [Goldstein and Ritter, 2003, p. 121].

In the introduction, Einstein wrote that it had occurred to him in the meantime that the structure of distant parallelism allows the identification of the gravitational and electromagnetic field equations in a most natural manner. He specialized to the case of four dimensions and identified the electromagnetic potential with the quantity

$$\phi_\mu \equiv \Lambda_{\mu\alpha}^\alpha = \frac{1}{2} h^{av} (h_{a\mu,v} - h_{av,\mu}). \quad (12)$$

More precisely, he stated that  $\phi_\mu = 0$  would be the mathematical expression for the absence of any electromagnetic field. But he added in a footnote that the same could be expressed by the condition  $\phi_{(\mu,v)} = 0$  and observed that this fact would result in a “certain indeterminateness of the interpretation” (“gewisse Unbestimmtheit der Deutung”).

The field equations are now given by specifying the Lagrangian  $\mathcal{H}$  as

$$\mathcal{H} = h g^{\mu\nu} \Lambda_{\mu\beta}^\alpha \Lambda_{\nu\alpha}^\beta \quad (13)$$

$$= \frac{1}{4} h h_\mu^a h_{av} h^{b\alpha} (h_{b\mu,\beta} - h_{b\beta,\mu}) h^{c\beta} (h_{c\nu,\alpha} - h_{c\alpha,\nu}). \quad (14)$$

In linear approximation,  $h_{a\mu} = \delta_{a\mu} + \tilde{h}_{a\mu}$ ,  $|\tilde{h}_{a\mu}|, |\partial\tilde{h}_{a\mu}| \ll 1$ , Einstein obtained the field equations explicitly as

$$\tilde{h}_{\beta\alpha,\mu\mu} - \tilde{h}_{\mu\alpha,\mu\beta} + \tilde{h}_{\alpha\mu,\mu\beta} - \tilde{h}_{\beta\mu,\mu\alpha} = 0. \quad (15)$$

Introducing the metric field in first approximation as

$$\tilde{g}_{\alpha\beta} = \delta_{\mu\nu} + \tilde{h}_{\alpha\beta} + \tilde{h}_{\beta\alpha} \quad (16)$$

and the electromagnetic four-potential  $\tilde{\phi}_a$  as

$$\tilde{\phi}_a = \frac{1}{2} (\tilde{h}_{\alpha,\mu}^\mu - \tilde{h}_{\mu,\alpha}^\mu), \quad (17)$$

the linearized field equations (15) turn into

$$\frac{1}{2} (-\tilde{g}_{\beta\alpha,\mu\mu} - \tilde{g}_{\mu\alpha,\mu\beta} + \tilde{g}_{\alpha\mu,\mu\beta} - \tilde{g}_{\beta\mu,\mu\alpha}) = \tilde{\phi}_{\alpha,\beta} - \tilde{\phi}_{\beta,\alpha}. \quad (18)$$

Since the absence of any electromagnetic field was expressed by  $\phi_\mu \equiv 0$ , Eq. (18) then turns into the linear approximation of the Ricci tensor  $R_{\alpha\beta}$ , given in terms of the metric, just as in standard general relativity.

The vacuum Maxwell equations are recovered in this approximation by taking the divergence of  $\tilde{\phi}_\alpha$ , which vanishes on account of (18) contracted over  $\alpha$  and  $\beta$ , which gives

$$\tilde{\phi}_{\alpha,\alpha} = 0, \quad (19)$$

and by

$$\tilde{\phi}_{\alpha,\beta\beta} = 0, \quad (20)$$

which follows from the fact that the left-hand side  $L_{\alpha\beta}$  of (18) satisfies the identity

$$\left( L_{\alpha\beta} - \frac{1}{2} \delta_{\alpha\beta} L_{\sigma\sigma} \right)_{,\beta} = 0. \quad (21)$$

Equations (19) and (20) together imply the vanishing of the divergence of the electromagnetic field  $\phi_{\mu,v} - \phi_{v,\mu}$ , which is just the inhomogeneous set of Maxwell equations in the absence of an external current. The homogeneous Maxwell equations are, of course, trivially fulfilled if an electromagnetic potential exists.

In a note added at proof stage, he observed that quite similar results could be obtained for the Lagrangian

$$\mathcal{H} = h g_{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} \Lambda_{\alpha\beta}^{\mu} \Lambda_{\sigma\tau}^{\nu} \quad (22)$$

and concluded that there is an ambiguity in the choice of  $\mathcal{H}$ .

In summary, Einstein had given, in his first two notes, an exposition of a certain generalization of Riemannian geometry. Making use of the concept of tetrad fields, he had introduced a geometry with vanishing Riemann–Cartan curvature but nonvanishing torsion. He had also realized that this framework opened up new vistas for the realization of a unified field theory and had proposed a first field equation for the tetrads by stipulating the simplest Lagrangian for a variational formulation.

### 3. Interaction with others

Einstein's first two notes on teleparallelism appear to be conceived and composed without any interaction with other mathematicians or physicists. This is confirmed by Einstein explicitly.

After twelve years of searching with many disappointments I now discovered a metric continuum structure that lies between the Riemannian and the Euclidean structures and the elaboration of which leads to a truly unified field theory.<sup>27</sup>

Nor does Einstein acknowledge any relevant literature in those first two notes. The only reference to existing work in the field that he did give concerned a—problematic, as we have seen, see Section 2.1—comparison of the *Fernparallelismus* approach with the standard Riemannian geometry and with Weyl's *Nahegeometrie*. But this reference is, in fact, too vague to be counted as a real citation.

Soon after the publication of Einstein's first two notes this situation changed. Einstein entered into intense interaction with several other mathematicians and scientists. He began a collaboration with the

<sup>27</sup> “Nach zwölf Jahren enttäuschungsreichen Suchens entdeckte ich nun eine metrische Kontinuumstruktur, welche zwischen der Riemannschen und der Euklidischen liegt, und deren Ausarbeitung zu einer wirklich einheitlichen Feldtheorie führt” [Einstein, 1929a, p. 130].

mathematician Herman Müntz on special solutions of the theory. He was alerted to earlier pertinent work in the mathematics literature by Roland Weitzenböck. And later in the year, Cornelius Lanczos joined Einstein in Berlin on a year of absence from Frankfurt. He also acknowledged contributions by Jakob Grommer who had been working with him in Berlin all the time. Interactions with Elie Cartan and Walther Mayer were also important but will be discussed later on since they began much later.

### 3.1. *The correspondence with Herman Müntz*

Chaim Herman Müntz (1884–1956) had studied mathematics in Berlin and had obtained his Ph.D. in 1910 with a thesis on the partial differential equations of the minimal surface.<sup>28</sup> The existing correspondence between Einstein and Müntz in the Einstein Archives suggest that Müntz and Einstein had contact already before summer 1928. Müntz was living in Berlin at the time, and according to Ortiz and Pinkus [Ortiz and Pinkus, 2005] he may have been working as Einstein's scientific collaborator as early as summer 1927.

Their extensive correspondence about teleparallelism appears to have been triggered by a letter from Müntz in which he pointed out that the field equations in first approximations are fully integrable.<sup>29</sup> In the sequel, Müntz was concerned with the task of computing the special case of spatial spherical symmetry. The correspondence shows that Einstein kept Müntz informed about his considerations regarding the proper field equations, asking him about explicit calculations for each new version of them. These calculations are acknowledged in [Einstein, 1929a, p. 132] and in [Einstein, 1929b, p. 7]. Müntz was also credited with pointing out the problem of compatibility of the field equations derived in [Einstein, 1929b]; see [Einstein, 1929e, p. 156]. In fact, in a letter, dated 18 March 1929 (EA 18-335), Müntz suggested rewriting an earlier version of the introduction of [Einstein, 1929e]. Their collaboration ended some time in 1929 when Müntz accepted a call as professor of mathematics at the university of Leningrad.

### 3.2. *The correspondence with Roland Weitzenböck*

A few days after Einstein had learnt from Müntz about the possibility of finding explicit solutions for his equations in first approximation, he received further correspondence regarding his new theory. On August 1, 1929, Einstein received a letter saying

The connection components that you denote [...] by  $\Delta_{\mu\sigma}^{\nu}$  were published first (1921) in my encyclopedia article III E 1 in note 59 with No. 18; more explicitly in my *Invariant Theory* (1923) (Groningen, Noordhoff), p. 317ff.<sup>30</sup>

The author was Roland Weitzenböck (1885–1955), who had been appointed professor of mathematics at the University of Amsterdam in 1921 at the initiative of Brouwer [van Dalen, 1999, Section 9.4]. The references are to [Weitzenböck, 1921, 1923].

<sup>28</sup> For biographical information on Müntz, see [Pais, 1982, pp. 491ff; Ortiz and Pinkus, 2005].

<sup>29</sup> “Es handelt sich darum, dass man die ersten Näherungsgleichungen [...] vollständig integrieren kann” [Einstein to Müntz, 26 July 1928, EA 18-328].

<sup>30</sup> “Die von Ihnen [...]  $\Delta_{\mu\sigma}^{\nu}$  genannten Zusammenhangskomponenten finden sich zuerst (1921) in meinem Enzyklopädie-Artikel III E 1 in Anmerkung 59 bei No. 18; ausführlicher in meiner *Invariantentheorie* (1923) (Groningen, Noordhoff), p. 317ff” [Weitzenböck to Einstein, 1 August, 1929, EA 23-367].

Weitzenböck also listed some later papers by G. Vitali, G.F.C. Griss, M. Euwe, E. Bortolotti, and L.P. Eisenhart<sup>31</sup> that would deal with the issue of parallel transport and differential invariants in manifolds endowed with an  $n$ -Bein-field.

More specifically, Weitzenböck stated a formal result relevant for Einstein's attempts to derive the field equations on the basis of a variational formulation. He claimed that any Lagrangian, i.e., any function that is invariant under both general coordinate transformations *and* rotations of the tetrads, can be built up from  $h = |h_{\alpha\nu}|$ ,  $g_{\mu\nu}$ ,  $g^{\mu\nu}$ , and  $\Lambda_{\alpha\beta}^{\nu}$  and its covariant derivatives with respect to the connection  $\Delta_{\alpha\beta}^{\nu}$ . Moreover, he stated the proposition that  $h$  is the only such function of order zero,<sup>32</sup> no function of first order exists that is linear in  $\Lambda_{\alpha\beta}^{\nu}$ , and any function of first order that is quadratic in  $\Lambda_{\alpha\beta}^{\nu}$  is built up of three invariants (see Eqs. (24)–(26) below). Incidentally, these quantities are sometimes referred to as Weitzenböck invariants in the modern literature. He announced that he was going to write a short communication about these results and asked whether Einstein would be willing to present such a note to the Prussian Academy for publication in its proceedings.

Einstein was quick to respond on 3 August, two days later. He apologized saying that he had written the first two notes while lying in bed with a “severe heart problem” and pointed out that he had asked Planck to inquire from the mathematicians in the Academy whether such notions are in fact known to the mathematicians. But Planck, he continued, had told him that a publication would be justified already from the physics point of view. Therefore he, Einstein, had agreed that Planck should proceed and submit his manuscript for publication as is. Of course, he would be all in favor of publishing a note by Weitzenböck.

Einstein added that he had in the meantime lost some confidence in the theory. While the quantities  $\phi_{\mu} = \Lambda_{\mu\alpha}^{\alpha}$  would satisfy Maxwell's equations, one would not, conversely, have a corresponding tetrad field for any solution of the Maxwell equations. In particular, a spherically symmetric electric field seemed not to exist in the new theory.

Weitzenböck sent his note without further delay on August 8. In his letter, he also asked a couple of questions about Einstein's second note. One point concerned the Einstein's approximation procedure and was clarified to be due to the fact that in setting  $h_{a\mu} = \delta_{a\mu} + \tilde{h}_{a\mu}$  Einstein had also, but only tacitly, assumed that the derivatives  $\partial \tilde{h}_{a\mu}$  would be of first order as well. The second point concerned the question of how to recover the vacuum field equation of the old theory of general relativity from the Weitzenböck invariants.

In his response, Einstein explained his approximation procedure.<sup>33</sup> He did not respond to Weitzenböck's second point of recovering the old gravitational equations<sup>34</sup> but he reiterated his new doubts with respect to the viability of the theory since it did not readily allow for the existence of electrically charged particle-like solutions. But he added:

<sup>31</sup> All references given in the letter are included in the more complete list given in [Weitzenböck, 1928, p. 466].

<sup>32</sup> The order of the function is defined to be the highest order of differentiation in its arguments, see [Weitzenböck, 1928, p. 470].

<sup>33</sup> Einstein somewhat missed, however, Weitzenböck's point: “Ich kann nicht begreifen, was Sie an meiner diesbezüglichen einfachen Rechnung auszusetzen haben” [Einstein to Weitzenböck, 16 August 1928, Centre for Mathematics and Computer Science (Amsterdam), library]. It was Weitzenböck himself who gave the answer to his own question in his response letter.

<sup>34</sup> That point was addressed later in a letter by Lanczos, see the discussion below.

But one has to be careful with a definite judgement since the limits of validity of the Maxwellian equations is an unsolved problem.<sup>35</sup>

He continued with an interesting heuristic comment indicating that he would be prepared to call into question other aspects of his heuristics if this should be necessary.

In any case, the combination of an integrable parallel transport with a metric seems to me very natural since already the assumption of a metric in a single point of the continuum overdetermines the metric if the law of parallel transport is given. But the metric need not be defined by a quadratic function. However, this is made probable by the principle of the constancy of the velocity of light.<sup>36</sup>

Einstein promised to present Weitzenböck's note to the Academy on the very next occasion. Due to the summer break, the next meeting, however, took place only in October and Weitzenböck's note was indeed presented on October 18, and its published version was issued on 28 November 1928.

Einstein mentioned Weitzenböck in three of his next papers [Einstein, 1929a, 1929b, 1930a], and temporarily adopted his notation for the *n*-Beins (see Appendix A). But their correspondence seems to have ended at this point, and there is no indication that Weitzenböck and Einstein had any further interaction in this matter.

### 3.3. The cooperation with Jakob Grommer and Cornelius Lanczos

The epistolary exchange with the mathematicians Müntz and Weitzenböck had been triggered by the publication of Einstein's first two notes. Two other scientists were important for Einstein at this time, his long-standing assistant Jakob Grommer and the theoretical physicist Cornelius Lanczos.

Jakob Grommer (1879–1933) had been working with Einstein for several years.<sup>37</sup> In fact, in 1925 Einstein wrote that Grommer had “faithfully assisted me in recent years with all calculations in the area of general relativity theory.”<sup>38</sup> Their collaboration resulted in a number of joint publications. Regarding Grommer's role in the *Fernparallelismus* project, there are only a few extant letters since most of their interaction was in person. Grommer had voiced doubts about the equivalence of the electromagnetic equations obtained in linear approximation with Maxwell's equations in Einstein's first version of field equations.<sup>39</sup> Einstein acknowledged Grommer's assistance in [Einstein, 1929b] but did not specify his contribution.

Some time in 1929, Grommer seems to have gone to Minsk to accept a teaching position at the university. Possibly in an attempt to find a successor for Grommer, Einstein was eager to arrange for Cornelius Lanczos (1893–1974) to come to Berlin for a year. Lanczos was *Privatdozent* at the university of Frank-

<sup>35</sup> “Man muss aber mit einem endgültigen Urteil vorsichtig sein, da die Grenze der Gültigkeit der Maxwell'schen Gleichungen ja ein ungeklärtes Problem ist” [ibid].

<sup>36</sup> “Jedenfalls erscheint mir die Kombination einer integrablen Parallelverschiebung mit einer Metrik sehr natürlich, da schon die Annahme der Metrik in *einem* Punkte des Kontinuums die Metrik überbestimmt, wenn das Verschiebungsgesetz gegeben ist. Allerdings brauchte die Metrik nicht durch eine quadratische Funktion definiert zu sein, aber dafür spricht das Prinzip von der Konstanz der Lichtgeschwindigkeit” [ibid].

<sup>37</sup> For biographical information on Grommer, see [Pais, 1982, pp. 487f].

<sup>38</sup> Quoted [ibid].

<sup>39</sup> See [Einstein to Müntz, end of July 1928, EA 18-311].



furt and took a year of leave of absence in order to be able to work with Einstein in Berlin.<sup>40</sup> Lanczos started to work with Einstein in Berlin on November 1, 1928. The stay was supported by a grant from the *Notgemeinschaft Deutscher Wissenschaft*. Einstein thanked Lanczos in the introduction of [Einstein, 1929e] for pointing out a problem with the compatibility of the field equations in that note. Lanczos also found out that the Lagrangian advanced in [Einstein, 1929e] is equivalent to the usual Riemann scalar (see the discussion below in Section 5.1). Lanczos himself also published a little semipopular note on the *Fernparallelismus* theory [Lanczos, 1929] in July 1929 and a more extended but also nontechnical account in 1931 [Lanczos, 1931].

## 4. Searching for field equations

### 4.1. Einstein's next papers

Einstein's further progress and the interaction with the aforementioned mathematicians is reflected in a semipopular overview of the present state of field theory, two further notes on the subject in the *Sitzungsberichte*, and two newspaper articles.

Soon after Weitzenböck's note appeared in late November, Einstein had a chance to react to it in print. In early November 1928, he had been asked to contribute to a *Festschrift* on the occasion of the 70th birthday of Aurel Stodola, professor of mechanical engineering at Zurich's polytechnic. That birthday would take place on May 10, 1929, but the *Festschrift* was to be completed ahead of time. Einstein agreed to contribute a semipopular review article "On the Present State of Field Theory" [Einstein, 1929a]. The manuscript for this paper was submitted on 10 December 1928.<sup>41</sup>

At the end of this more general survey of the history of field theory, Einstein briefly sketched his new approach, commenting also on the derivation of field equations. He mentioned calculations of the equations of motion for chargeless particles, undertaken together with Müntz. With reference to Weitzenböck, Einstein introduced a change of notation: algebraic indices are now written to the left (see Appendix A).

In a final paragraph that appears to have been added to this paper at proof stage, Einstein remarked that he had in the meantime convinced himself that field equations for the theory were not obtained by a variational principle but obtained by other considerations.

The following paper again appeared in the Academy's *Sitzungsberichte* and was presented to the Academy for publication on January 10 [Einstein, 1929b]. It indeed advanced a new derivation of field equations that did not make use of a Hamiltonian principle. In the paper Einstein also introduced a few new notational conventions.<sup>42</sup>

The reception of this paper by the public should remind us that nothing Einstein did at the time took place in an ivory tower.<sup>43</sup> Albrecht Fölsing gives a vivid account of the immense public interest in Einstein's new theory.<sup>44</sup> The January paper itself was printed and reprinted several times by the Prussian Academy with a record number of copies. The public interest in Einstein's new field theory is exem-

<sup>40</sup> For more biographical information on Lanczos and an account of his interactions with Einstein, see [Stachel, 1994].

<sup>41</sup> See marginal notes on EA 22-261 and EA 22-262.

<sup>42</sup> The notational idiosyncrasies associated with Einstein's *Fernparallelismus* approach are summarized in Appendix A.

<sup>43</sup> For a perceptive discussion of Einstein's iconic image in the public, see [Friedman and Donley, 1985].

<sup>44</sup> [Fölsing, 1997, pp. 604ff]; see also [Pais, 1982, p. 346].

plified by the following quote from a letter by Eddington who was acknowledging receipt of copies of Einstein's recent papers, among them [Einstein, 1929b]:

You may be amused to hear that one of our great Department Stores (Selfridges) has pasted up in its window your paper (the six pages pasted up side by side) so that passers by can read it all through. Large crowds gather round to read it!<sup>45</sup>

The craze apparently had begun with an article in the *New York Times* of 4 November 1928 under the title “Einstein on Verge of Great Discovery; Resents Intrusion.” The author of this article, Paul D. Miller, gave an account of how he had succeeded in visiting Einstein in his Berlin home. It is a striking example of grooming the myth of this mysteriously creative genius. The sick Einstein supposedly “sat on a sunny beach and appeased his desire to work by playing his violin to the waves” but then came up with a new theory that “will startle the world far more than relativity did.” The article, in any case, seems to have triggered the interest of numerous other journalists in Einstein's new work.

The journalists, thus alerted of those great events in science, may have been all too glad to learn that, in early January, another publication on this new theory appeared and warranted press coverage. In any case, on January 12, two days after the submission of [Einstein, 1929b] to the Academy, the front page of the *New York Times* again informed their readers that “Einstein Extends Relativity Theory.” The subtitle: “‘Book,’ Consisting of Only Five Pages, Took Berlin Scientist Ten Years to Prepare” may help to explain why the management of Selfridges came up with the idea of attracting the curiosity of possible clients by putting up a copy of this marvel in their window. An English translation of the note, including all formulas, appeared on the title page of the *New York Herald Tribune* on February 1. And in response to the overwhelming public interest in his new theory, Einstein published two popular and nontechnical accounts of the latest developments in the *New York Times* on February 3 [Einstein, 1929c] and in the *London Times* of February 4 [Einstein, 1929d].

The essays are a *tour de force* through the history of field theory. At their very end, Einstein gave a characterization of distant parallelism by illustrating the effect of torsion. He has the reader consider two parallel lines  $E_1L_1$  and  $E_2L_2$  and on each a point  $P_1$ , respectively  $P_2$ . On the first line,  $E_1L_1$ , one now chooses another point  $Q_1$ . Torsion is then expressed by the fact that parallelograms do not close.

If we now draw through  $Q_1$  a straight line  $Q_1-R$  parallel to the straight line  $P_1, P_2$ , then in Euclidean geometry this will cut the straight line  $E_2L_2$ ; in the geometry now used the line  $Q_1-R$  and the line  $E_2L_2$  do not in general cut one another. [Einstein, 1929c]

Einstein added

To this extent the geometry now used is not only a specialization of the Riemannian but also a generalization of the Euclidean geometry. [ibid]

In the final paragraph he then stated the expectation that the solution to the mathematical problem of the correct field laws would be given by “the simplest and most natural conditions to which a continuum of this kind can be subjected.” Einstein concluded that

<sup>45</sup> [Eddington to Einstein, 11 February, 1929, EA 9-292].

the answer to this question which I have attempted to give in a new paper yields unitary field laws for gravitation and electromagnetism. [ibid]

The generic title of the January paper in the *Sitzungsberichte* (“On the Unified Field Theory”) may have helped to deceive the public about the real content of this rather specific and technical communication. The title of the next paper on the *Fernparallelismus* approach would surely have been less attractive for a general public. It is entitled “Unified Field Theory and the Hamiltonian Principle” [Einstein, 1929e]. It addressed an objection raised by Lanczos and Müntz. They had objected that the compatibility of the field equations of the previous note was not established because of the failure to identify four identical relations among them. Einstein now returned to the variational approach and gave a Hamiltonian formulation of the field equations which thus would also guarantee their compatibility.

#### 4.2. The field equations

Let us now take a closer look at the problem of finding and justifying field equations within the teleparallel framework. The tetrad field  $h_{s\mu}$  defines both the metric tensor field  $g_{\mu\nu}$ , see Eq. (3), and the electromagnetic vector potential  $\phi_\mu$ , see Eq. (12). Its 16 components are the dynamical variables of the theory. The fundamental question therefore arises as to the field equations that determine the tetrad field. Einstein had first discussed this question in his second note of June 14, 1928, but doubts were raised in the sequel about the correct field equations and their derivation. These doubts remained alive with Einstein until the very end of the *Fernparallelismus* episode and are also the major reason for his eventually giving up the teleparallel approach.

We will here review the early attempts at finding field equations and their derivations as put forward by Einstein in the course of elaborating the implications of distant parallelism. A closer analysis of the chronology reveals that Einstein wavered between two distinct approaches to finding, deriving, and justifying field equations. In one approach, he was starting from a variational principle and was looking for the correct Lagrangian. In another approach, he was trying to find a set of overdetermined field equations plus a number of mathematical identities.

The existence of two distinct approaches is strongly reminiscent of the heuristics followed for the *Entwurf* theory; see Section 7 below. And as was the case with the reconstruction of the genesis and demise of the *Entwurf*, the dynamics of going from one approach to the other, it seems to me, can only be reconstructed with some confidence on the basis of more information taken from contemporary correspondence and research manuscripts. The following sketch will therefore necessarily have a preliminary character.

##### 4.2.1. The variational approach

The field equations advanced in Einstein’s second note on the distant parallelism approach were defined by demanding that the variation of a scalar and globally Lorentz-invariant action integral  $\int \mathcal{H} d\tau$  with respect to the components of the tetrad field  $h_{a\mu}$  vanish; see Eq. (11) above. The Lagrangian  $\mathcal{H}$  entering the action integral had been given in terms of the invariant  $g^{\mu\nu} \Lambda_{\mu\beta}^\alpha \Lambda_{\nu\alpha}^\beta$  as in Eq. (13).

Einstein did not give any motivation for this kind of Lagrangian. But it would be a natural *Ansatz* or candidate field equation for him to try. The torsion tensor  $\Lambda_{\mu\nu}^\alpha$  was the crucial new quantity of the theory and the invariant was the simplest combination that was invariant both for general coordinate transformations and for rotation of the tetrads. But the torsion tensor allows for different ways to form

a scalar expression. Let us recall then that in a little note added in proof, Einstein already observed that “similar results are obtained” on the basis of the Lagrangian  $\mathcal{H} = h g_{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} \Lambda_{\alpha\beta}^{\mu} \Lambda_{\sigma\tau}^{\nu}$ ; see (22) above. Einstein commented that “for the time being” there was an uncertainty regarding the choice of  $\mathcal{H}$ . It is unclear whether there was an external trigger for this realization.

But things got worse. In his contribution to the Stodola-Festschrift Einstein briefly sketched his new approach. With respect to the derivation of field equations, Einstein now considered a generic Lagrangian

$$\mathcal{H} = h(AJ_1 + BJ_2 + CJ_3) \quad (23)$$

where

$$J_1 = g^{\mu\nu} \Lambda_{\mu\beta}^{\alpha} \Lambda_{\nu\alpha}^{\beta}, \quad (24)$$

$$J_2 = g^{\mu\nu} \Lambda_{\mu\alpha}^{\alpha} \Lambda_{\nu\beta}^{\beta}, \quad (25)$$

$$J_3 = g^{\mu\sigma} g^{\nu\tau} g_{\lambda\rho} \Lambda_{\mu\nu}^{\lambda} \Lambda_{\sigma\tau}^{\rho}. \quad (26)$$

Einstein does not explicitly refer to Weitzenböck’s paper [Weitzenböck, 1928] in the Stodola-Festschrift in this context, although he does mention his name with respect to a change of notation for the tetrads. It should be pointed out that the three terms  $J_i$  that are explicitly listed in the Stodola-Festschrift (p. 470) as the invariants from which a Lagrangian density will have to be constructed are, in fact, the only invariants (under both general coordinate transformations and rotations of the tetrads) of second degree in  $\Lambda_{\alpha\beta}^{\nu}$ , as was shown by Weitzenböck.

Einstein remarked that

The elaboration and physical interpretation of the theory is made difficult by the lack of an apriori constraint for choosing the ratio of the constants  $A, B, C$ .<sup>46</sup>

Einstein’s first *Ansatz* Eq. (13) of the June 14 note is contained in the generic Lagrangian (23) by specifying  $A = 1, B = C = 0$ . The alternative Lagrangian (22) that was advanced in a note added in proof to that June 14 paper would be given by specifying the case of  $C = 1, A = B = 0$ . In the Stodola-Festschrift, Einstein then specified the case  $B = -A, C = 0$ , which would read explicitly

$$\mathcal{H} = h(g^{\mu\nu} \Lambda_{\mu\alpha}^{\alpha} \Lambda_{\nu\beta}^{\beta} - g^{\mu\nu} \Lambda_{\mu\beta}^{\alpha} \Lambda_{\nu\alpha}^{\beta}). \quad (27)$$

He observed, however, that the specialization  $B = -A, C = 0$  should be taken only at the level of the field equations, not at the level of the variational principle. Otherwise, the electromagnetic field equations would not be obtained. This arcane remark is not further explained by explicit calculations.

Moreover, in the last paragraph of the Stodola-Festschrift, apparently added at proof stage, Einstein remarked that he had in the meantime convinced himself that the “most natural” *Ansätze* for the field

<sup>46</sup> “Die Ausarbeitung und physikalische Interpretation der Theorie wird dadurch erschwert, dass für die Wahl des Verhältnisses der Konstanten  $A, B, C$  a priori keine Bindung vorhanden ist” [Einstein, 1929a, p. 132].

equations are not obtained on the basis of a Hamiltonian principle.<sup>47</sup> For a different approach, he referred to his new paper “On the Unified Field Theory” in the Prussian Academy proceedings [Einstein, 1929b].

However, that alternative approach of January 10, which will be discussed below, was shortlived. Already some two months later, on 21 March 1929, Einstein returned to the variational approach for deriving the field equations.

Since Einstein had introduced in the January 10 paper a number of new conventions, the notation used in the March note is slightly different from the notation used in the Stodola-Festschrift. Thus, he had dropped a factor of  $1/2$  in the definition of the torsion and he had introduced an idiosyncratic convention of indicating raising and lowering indices by underlining them. He also used a slightly different notation for the terms defined in Eqs. (24), (25), (26) using  $\mathcal{J} = hJ$ , and he renumbered two terms; i.e., he has  $\mathcal{J}_2 \equiv hJ_3$  respectively  $\mathcal{J}_3 \equiv hJ_2$ . If we keep with the notation of the Stodola-Festschrift (23), (24)–(26), Einstein now advanced the following Lagrangian (up to an overall constant):

$$\mathcal{H} = h \left( \frac{1}{2} J_1 - J_2 + \frac{1}{4} J_3 \right). \quad (28)$$

This Lagrangian is explicitly justified by the following two postulates.  $\mathcal{H}$  must be a function of second degree in the torsion tensor  $\Lambda_{\mu\nu}^\alpha$  which makes it a linear combination of the three terms  $J_1$ ,  $J_2$ ,  $J_3$ .<sup>48</sup> Second, the resulting field equation must be symmetric in the free indices, and Einstein claimed that this postulate uniquely fixes the specific linear combination (28).

More specifically, Einstein claimed that the combination (28) produces only one part of the field equations, i.e., the part that reduces to the gravitational field equation in linear approximation. In order to obtain the electromagnetic field equations, he proposed to consider a slightly modified Lagrangian,

$$\bar{\mathcal{H}} = \mathcal{H} + h\epsilon_1 \left( \frac{1}{2} J_1 - \frac{1}{4} J_2 \right) - h\epsilon_2 J_3, \quad (29)$$

where the existence of electric charges demands taking the limit  $\epsilon_2/\epsilon_1 \rightarrow 0$ . In that limit, the relation

$$S_{\underline{\mu}\underline{\nu}}^\alpha = 0 \quad (30)$$

is obtained, where the quantity  $S_{\underline{\mu}\underline{\nu}}^\alpha$  was defined as the completely antisymmetrized torsion

$$S_{\underline{\mu}\underline{\nu}}^\alpha = \Lambda_{\underline{\mu}\underline{\nu}}^\alpha + \Lambda_{\alpha\underline{\mu}}^\nu + \Lambda_{\nu\alpha}^\underline{\mu}, \quad (31)$$

using Einstein’s temporary convention to indicate a raising respectively lowering of an index by underlining; see (A.4) below. Einstein claimed that the relation (30) implies that the combination (28) is equivalent to the earlier combination  $J_1 - J_2$  of [Einstein, 1929a].

<sup>47</sup> “Inzwischen hat mich eine tiefere Analyse der allgemeinen Eigenschaften der Strukturen der oben entwickelten Art zu der Überzeugung geführt, dass die natürlichsten Ansätze für die Feldgleichungen nicht aus einem Hamilton-Prinzip, sondern auf anderem Wege zu gewinnen sind” [Einstein, 1929a, p. 132].

<sup>48</sup> This fact is stated in [Weitzenböck, 1928, p. 470]; see the discussion in Section 3.2.

The procedure of varying a slightly modified Lagrangian in order to obtain the electromagnetic field equation had been developed partly within the overdetermination approach. The details were not, however, spelled out explicitly in the published papers on the *Fernparallelismus* approach.

In summary, Einstein had advanced four different field equations in three papers, which are given by the generic Lagrangian (23) and the following coefficients:

| Paper                         | Date             | A   | B  | C   |
|-------------------------------|------------------|-----|----|-----|
| [Einstein, 1928b]             | 14 June 28       | 1   | 0  | 0   |
| [Einstein, 1928b, note added] | after 14 June 28 | 0   | 0  | 1   |
| [Einstein, 1929a]             | 10 December 28   | 1   | −1 | 0   |
| [Einstein, 1929b]             | 21 March 29      | 1/2 | −1 | 1/4 |

We shall now turn to the second approach of deriving field equations for the teleparallel theory.

#### 4.3. The overdetermination approach

Already by the end of 1928, around the time when he had submitted his paper for the Stodola-Festschrift, Einstein may have become dissatisfied with the variational approach, possibly because he had not succeeded in finding a convincing way of getting unique field equations. But there were also other difficulties associated with the demands that the electromagnetic field equations should be obtained in the linearized approximation and that nonsingular, spherically symmetric, and stationary, charged, or massive solutions to the field equations should exist.<sup>49</sup>

In any case, a few days after sending off his manuscript for the Stodola Festschrift, he wrote to Hermann Müntz

I have had a simple, cheeky idea which throws the Hamiltonian principle overboard. The cart shall now be put before the horse: I choose the field equations in such a way that I am sure that they imply the Maxwellian equations.<sup>50</sup>

The idea was to use an identity that implies the validity of the Maxwell equations and construct field equations by the demand that this identity was automatically satisfied. But again Einstein encountered technical difficulties showing him that the simple idea was not feasible.

The derivation of the field equations by means of the identity is a task that is more subtle than I originally thought.<sup>51</sup>

<sup>49</sup> For a historical discussion of the significance of the requirement of nonsingular particle solutions, see [Earman and Eisenstaedt, 1999].

<sup>50</sup> “Ich habe eine einfache, freche Idee gehabt, die das Hamilton’sche Prinzip über Bord wirft. Das Pferd soll nun vom Schwanz aus aufgezäumt werden: ich wähle die Feldgleichungen so, dass ich sicher bin, dass sie die Maxwellschen Gleichungen zur Folge haben” [Einstein to Müntz, 13 December 1928, EA 18-317].

<sup>51</sup> “Die Aufstellung der Feldgleichungen mit Hilfe der Identität ist eine subtilere Aufgabe, als ich ursprünglich dachte” [Einstein to Müntz, 15 December 1928, EA 18-318].



However, he did pursue the general approach further and soon came up with another derivation of the field equations.

The next paper then was the January note that would attract so much public interest [Einstein, 1929b]. It presented a different approach to a derivation of field equations since that derivation on the basis of a Hamiltonian principle had not “led to a simple and completely unique path.”<sup>52</sup>

Einstein now argued like this. He first derived two sets of identities for the torsion tensor  $\Lambda_{\mu\nu}^\alpha = \Delta_{\mu\nu}^\alpha - \Delta_{\nu\mu}^\alpha$ . The first identity was obtained by starting from the vanishing of the Riemann curvature (7) for the Weitzenböck connection (6). If (7) is cyclically permuted in the lower indices and added it produces the identity

$$0 \equiv \Lambda_{\kappa\lambda,\mu}^l + \Lambda_{\lambda\mu,\kappa}^l + \Lambda_{\mu\kappa,\lambda}^l + \Delta_{\sigma\kappa}^l \Lambda_{\lambda\mu}^\sigma + \Delta_{\sigma\lambda}^l \Lambda_{\mu\kappa}^\sigma + \Delta_{\sigma\mu}^l \Lambda_{\kappa\lambda}^\sigma, \quad (32)$$

which can be rewritten using covariant derivatives (with respect to the connection  $\Delta_{\sigma\kappa}^l$ ),

$$\Lambda_{\kappa\lambda;\mu}^l = \Lambda_{\kappa\lambda,\mu}^l + \Lambda_{\kappa\lambda}^\sigma \Delta_{\sigma\mu}^l - \Lambda_{\sigma\lambda}^l \Delta_{\kappa\mu}^\sigma - \Lambda_{\kappa\sigma}^l \Delta_{\lambda\mu}^\sigma, \quad (33)$$

as

$$0 \equiv \Lambda_{\kappa\lambda;\mu}^l + \Lambda_{\lambda\mu;\kappa}^l + \Lambda_{\mu\kappa;\lambda}^l + \Lambda_{\kappa\alpha}^l \Lambda_{\lambda\mu}^\alpha + \Lambda_{\lambda\alpha}^l \Lambda_{\mu\kappa}^\alpha + \Lambda_{\mu\alpha}^l \Lambda_{\kappa\lambda}^\alpha. \quad (34)$$

Contracting (34) and using  $\phi_\mu \equiv \Lambda_{\mu\alpha}^\alpha$ , see (12), the identity can be written as

$$0 \equiv \Lambda_{\kappa\lambda;\alpha}^\alpha + \phi_{\lambda;\kappa} - \phi_{\kappa;\lambda} - \phi_\alpha \Lambda_{\kappa\lambda}^\alpha, \quad (35)$$

and, introducing the tensor density

$$\mathfrak{B}_{\kappa\lambda}^\alpha = h(\Lambda_{\kappa\lambda}^\alpha + \phi_\lambda \delta_\kappa^\alpha - \phi_\kappa \delta_\lambda^\alpha), \quad (36)$$

(35) can further be rewritten as

$$\mathfrak{B}_{\kappa\lambda;\alpha}^\alpha - \mathfrak{B}_{\kappa\lambda}^\sigma \Lambda_{\sigma\alpha}^\alpha \equiv \mathfrak{B}_{\kappa\lambda/\alpha}^\alpha \equiv 0, \quad (37)$$

i.e., as the vanishing of some special divergence denoted by  $\dots/\alpha$ . This notation shall be temporarily used here, too, to have a chance to see Einstein’s heuristics in his line of argument.

The second identity was derived by considering the commutator of the covariant derivatives for an arbitrary tensor  $T_{\dots}$ ,

$$T_{\dots;l;\kappa} - T_{\dots;\kappa;l} = -T_{\dots;\sigma} \Lambda_{l\kappa}^\sigma. \quad (38)$$

Inserting  $\mathfrak{B}_{\kappa\lambda}^\sigma$  for  $T_{\dots}$ , rewriting in terms of  $\dots/\alpha$ , and using (7) as well as the first identity (37), Einstein obtained the second identity as

$$(\mathfrak{B}_{\kappa\lambda/\lambda}^\alpha - \mathfrak{B}_{\kappa\tau}^\sigma \Lambda_{\sigma\tau}^\alpha)_{/\alpha} = 0. \quad (39)$$

<sup>52</sup> “[...] führte die Ableitung der Feldgleichung aus dem Hamiltonschen Prinzip auf keinen einfachen und völlig eindeutigen Weg. Diese Schwierigkeiten verdichteten sich bei genauerer Überlegung” [Einstein, 1929b, p. 2].

Field equations are now derived as follows. Identity (37) motivated Einstein to consider the vanishing of “the other divergence,” i.e.,

$$\mathfrak{B}_{\kappa\lambda/\lambda}^\alpha = 0, \quad (40)$$

as the field equation. In linear approximation, he obtained indeed the gravitational equations but could not get the electromagnetic equations, a difficulty that he traced back to the identity

$$\mathfrak{B}_{\kappa\lambda/\lambda/\alpha}^\alpha = \mathfrak{B}_{\kappa\lambda/\alpha/\lambda}^\alpha. \quad (41)$$

The trick to get the electromagnetic equations also was to look at the quantity

$$\overline{\mathfrak{B}}_{\kappa\lambda}^\alpha = \mathfrak{B}_{\kappa\lambda}^\alpha + \epsilon h(\phi_\lambda \delta_\kappa^\alpha - \phi_\kappa \delta_\lambda^\alpha) \quad (42)$$

and take as field equation

$$\overline{\mathfrak{B}}_{\kappa\lambda/\lambda}^\alpha = 0. \quad (43)$$

Maxwell’s equations would then be obtained by taking the divergence with respect to the index  $\alpha$ . The gravitational equations would still be obtained by taking the limit of  $\epsilon \rightarrow 0$ .

Going beyond the linear approximation, Einstein now started from the identity (39), and postulated the field equations

$$\overline{\mathfrak{B}}_{\kappa\lambda/\lambda}^\alpha - \overline{\mathfrak{B}}_{\kappa\tau}^\sigma \Lambda_{\sigma\tau}^\alpha = 0, \quad (44)$$

where again the electromagnetic equations are obtained by considering the divergence with respect to the index  $\alpha$  and the gravitational equations by taking the limit  $\epsilon \rightarrow 0$ . Consequently, the final field equations are

$$\mathfrak{B}_{\kappa\lambda/\lambda}^\alpha - \mathfrak{B}_{\kappa\tau}^\sigma \Lambda_{\sigma\tau}^\alpha = 0, \quad (45)$$

and

$$[h(\phi_{\kappa;\alpha} - \phi_{\alpha;\kappa})]_{/\alpha} = 0. \quad (46)$$

These are 20 equations for the 16 quantities  $h_{a\mu}$ . The compact notation involves the idiosyncratic notation of the divergence  $\dots_{/\alpha}$  introduced in (37), the convention of raising indices by underlining them according to (A.4), and the introduction of the quantities  $\mathfrak{B}_{\kappa\lambda}^\alpha$  in (36),  $\Lambda_{\mu\nu}^\alpha$  in (10), and  $\phi_\alpha$  in (12). Einstein argued that there were eight identities between these 20 equations. But he had explicitly given only four of them, i.e., (39). The problem here was that Einstein had erroneously assumed the existence of a set of identities compatible with the field equations, as pointed out to him soon by Lanczos and Müntz.

## 5. The *Mathematische Annalen* paper

The overdetermination approach had produced field equations (45) and (46) and the variational approach had produced the Lagrangian (29). It is unclear to me to what extent Einstein reflected on the compatibility of the two approaches, i.e., to what extent he tried to produce the same set of field equations along the two approaches, or specifically how the Lagrangian (29) published in March relates to the field equations (45) and (46) of January. In any case, it should have become clear that all explicit calculations in terms of the fundamental tetrad variables  $h_{a\mu}$  involved an appreciable amount of algebraic complexity, and it seems that many implications were only realized on the level of the linear approximation.

The theory of distant parallelism reached its mature stage in the summer of 1930 with Einstein's major publication concerning the *Fernparallelismus* approach: a review paper that was published in the *Mathematische Annalen* [Einstein, 1930a]. The publication history of this paper is a little involved and reflects an issue of priority that arose between Einstein and Elie Cartan. The paper also gave a new derivation of the final field equations along the overdetermination approach.

### 5.1. The publication history

The prehistory of this paper seems to begin with a letter by Elie Cartan that was sent to Einstein on 8 May 1929 and that triggered an extensive correspondence between the two scientists.<sup>53</sup> In this first letter, Cartan pointed out to Einstein that the mathematical framework of Einstein's *Fernparallelismus* was, indeed, a special case of a generalization of Riemannian geometry advanced by him in previous years

Now, the notion of Riemannian space endowed with a *Fernparallelismus* is a special case of a more general notion, that of a space with a Euclidean connection, which I outlined briefly in 1922 in an article in the *Comptes Rendues* [...] <sup>54</sup>

The reference is to [Cartan, 1922] and what Cartan here calls a Euclidean connection is a nonsymmetric linear connection on a real, differentiable manifold, thus allowing for both Riemannian curvature and torsion.<sup>55</sup> Moreover, Cartan pointed out that he had even spoken to Einstein about this generalized geometry when they had met, in 1922, at Hadamard's home. He even remembered that he had tried to illustrate the case of teleparallelism in his theory to Einstein on this occasion.

On receiving this letter, Einstein seems to have been quick to react. Apparently he sent off a review article of his theory to the *Zeitschrift der Physik* on the next day. This review article never appeared. In fact, it may have been sent off at the time prematurely only because Einstein, in his response to Cartan, another day later, wanted to mention this work of his. What he wrote to Cartan essentially was an acknowledgment that Cartan was right:

<sup>53</sup> This correspondence was published in [Debever, 1979].

<sup>54</sup> “Or la notion d'espace riemannien doué d'un *Fernparallelismus* est un cas particulier d'une notion plus générale, celle d'espace à connexion euclidienne, que j'ai indiquée succinctement en 1922 dans une note des *Comptes rendus*” [Cartan to Einstein, 8 May 1929] [Debever, 1979, Doc. I].

<sup>55</sup> See [Akivis and Rosenfeld, 1993, Chapter 7] for an account of Cartan's work on generalized spaces.

I see, indeed, that the manifolds used by me are a special case of those studied by you.<sup>56</sup>

By way of excuse, he pointed out that Weitzenböck had already written a review article on the mathematical foundations of teleparallelism with a supposedly complete bibliography but had failed to cite Cartan's work. And in his own review article of the previous day, he himself, so he wrote, had not mentioned any literature at all, not even his own papers.

But Einstein acknowledged Cartan's claim of priority and suggested that Cartan write a brief historical account, "a short analysis of the mathematical background," to be appended to his own paper but under Cartan's name.<sup>57</sup>

Cartan agreed in a letter of 15 May and, indeed, sent a manuscript to Einstein a little more than a week later, i.e., on May 24th.

One would think that Einstein, on receiving Cartan's manuscript would have forwarded it to the *Zeitschrift für Physik* as he had suggested to Cartan. It seems, however, that Einstein did not do so. What might have changed matters was perhaps a letter by Lanczos that Einstein may have received on the very same day, since the latter had written it the day before. In his correspondence, Lanczos communicated to Einstein his insight that in Weitzenböck's theory the scalar Riemannian curvature  $R$  is essentially equivalent to "the invariant preferred by you,"<sup>58</sup>  $\frac{1}{2}J_1 + \frac{1}{4}J_2 - J_3$ , plus a divergence. From this result, it clearly follows that the variational principle would not yield the electromagnetic equations.

In a letter to Müntz, written a few days later, Einstein wrote:

Regarding the whole problem Lanczos' discovery changes the situation profoundly.<sup>59</sup>

No paper on distant parallelism by Einstein or Cartan appeared in the *Zeitschrift für Physik*. Nevertheless, the next paper on teleparallelism by Einstein was a review paper and its aim was to present the theory in a self-contained way without reference to earlier publications. This work appeared in the *Mathematische Annalen* with a historical review paper on the subject by Cartan appended to it in the very same issue of this journal [Einstein, 1930a; Cartan, 1930].<sup>60</sup>

Einstein had been co-editor of the *Mathematische Annalen* from 1919 until 1928.<sup>61</sup> However, during that time he had published only a single paper in this journal himself [Einstein, 1927]. The paper on teleparallelism would be his only other paper published in the *Mathematische Annalen*.

<sup>56</sup> "Ich sehe in der That ein, dass die von mir benutzten Mannigfaltigkeiten in den von Ihnen studierten als Spezialfall enthalten sind" [Einstein to Cartan, 10 May 1929] [Debever, 1979, Doc. II].

<sup>57</sup> "Schreiben sie über diese mathematische Vorgeschichte eine kurze Charakteristik, die wir meiner neuen zusammenfassenden Arbeit anheften, natürlich unter Ihrem Namen, aber mit meiner Arbeit zu einem Ganzen vereinigt" [ibid]. One cannot help being reminded of Einstein's and Grossmann's earlier *Outline of a Generalized Theory of Relativity and of a Theory of Gravitation*. But in that case, Einstein and Grossmann had actually collaborated to obtain the results presented in their paper.

<sup>58</sup> "Die von Ihnen bevorzugte Invariante" [Lanczos to Einstein, 23 May 1929, EA 15-230].

<sup>59</sup> "Was das ganze Problem anlangt, ändert Lanczos Entdeckung die Situation von Grund aus" [Einstein to Müntz, 27 May 1929, EA 18-323]. In other words, what Lanczos had seen was that distant parallelism is essentially equivalent to classical general relativity, a fact that was fully realized only much later.

<sup>60</sup> This is another similarity to the Einstein–Grossmann collaboration. Both papers that were coauthored by Einstein and Grossmann appeared in the more mathematically oriented *Zeitschrift für Mathematik und Physik*, whereas Einstein published most of his other notes at the time in the *Annalen der Physik*.

<sup>61</sup> For a historical account of the changes in the editorial board of the *Mathematischen Annalen* in this period, see [van Dalen, 1990].

Einstein's paper in the *Annalen* is entitled “Unified Field Theory Based on the Riemann Metric and on Distant Parallelism” [Einstein, 1930a]. According to the published version it was received by the *Annalen* on 19 August 1929. But in a letter to the managing editor Otto Blumenthal, dated 19 August 1929, Einstein only announced submission of “an already completed summarizing work on the mathematical apparatus of the general field theory”<sup>62</sup> to the *Annalen*. In the letter, Einstein inquired whether “a treatise in the French language (ca. 12 pages long) on the prehistory of the problem”<sup>63</sup> composed by Cartan could be appended to his paper. He also enquired how long it would take until the paper would be printed.

A week later Einstein informed Cartan of the change regarding his publication plans and apologized for the long silence which was

caused by many doubts as to the correctness of the course I have adopted. But now I have come to the point that I am persuaded I have found the simplest legitimate characterization of a Riemannian metric with distant parallelism that can occur in physics.<sup>64</sup>

Einstein added that he now wanted to publish in the *Annalen* since “for the time being only the mathematical implications are explored and not their application to physics.”<sup>65</sup>

Blumenthal only responded on September 9 to Einstein's enquiry, agreeing to the proposal and informing Einstein that the publication would be delayed by approximately six months. A few days later, on 13 September, Einstein finally sent both manuscripts, his own and Cartan's, to Blumenthal for publication in the *Annalen*. In the covering letter, he expressed his understanding for the delay in publication but added

However, it is a pity because it delays the collaborative work of colleagues on this problem which is fundamental and, after the most recent results, really promising. Physics after all has a different rhythm than mathematics.<sup>66</sup>

Proofs of the paper were probably received by late November.<sup>67</sup> According to the title page of the pertinent issue of the *Annalen*, it was “completed” (“abgeschlossen”) on 18 December 1929.

<sup>62</sup> “Eine bereits fertiggestellte zusammenfassende Arbeit über den mathematischen Apparat der allgemeinen Feldtheorie” [Einstein to Blumenthal, 19 August 1929, EA 9-005].

<sup>63</sup> “Eine französisch geschriebene Abhandlung (etwa 12 Seiten Länge) über die Vorgeschichte des Problems” [ibid].

<sup>64</sup> “Verursacht durch viele Zweifel an der Richtigkeit des eingeschlagenen Weges. Nun aber bin ich soweit gekommen, dass ich die einfachste gesetzliche Charakterisierung einer Riemann-Metrik mit Fernparallelismus, welche für die Physik in Betracht kommen kann, gefunden zu haben überzeugt bin” [Einstein to Cartan, 25 August 1929] [Debever, 1979, Doc. V].

<sup>65</sup> “Einstweilen nur die mathematischen Zusammenhänge untersucht werden, nicht aber deren Anwendung auf die Physik” [ibid].

<sup>66</sup> “Es ist aber schade, weil die Mitarbeit der Kollegen an diesem fundamentalen und nach den letzten Ergebnissen wirklich aussichtsreichen Problem dadurch verzögert wird. Die Physik hat eben einen anderen Rhythmus als die Mathematik” [Einstein to Blumenthal, 13 September 1929, EA 9-009].

<sup>67</sup> In a letter to Einstein, dated 3 December, Cartan informed him that he had already returned the proofs which he had received “a few days ago” [Debever, 1979, Doc. VII].

## 5.2. The derivation of the field equations

Einstein's *Annalen* paper has five paragraphs. It begins with an exposition of the mathematical structure of *Fernparallelismus* in the first three paragraphs. Here he reverted to the original notation of writing both indices of the tetrads to the right, using Latin characters for algebra, Greek characters for coordinate indices. He also explicitly commented that he would no longer use the new divergence operation. In paragraphs four and five, he then discussed the field equations and their first approximation.

As pointed out explicitly in the introductory paragraph of the paper, the most important and in any case new part of the paper concerns the “derivation of the simplest field laws to which a Riemannian manifold with teleparallelism may be subjected.”<sup>68</sup> Here, however, he no longer proceeded along a variational approach but argued as follows.

Einstein observed that for the simplest field equations one is looking for conditions on the torsion tensor  $\Lambda_{\alpha\nu}^{\mu}$  expressed in terms of the Weitzenböck connection  $\Delta_{\alpha\nu}^{\mu}$ , respectively in terms of the tetrad fields  $h_{a\mu}$  (as in (10) but without the factor of  $1/2$ ). Although he does not say so explicitly in the paper, the rationale for this argument would be that for vanishing torsion one also has vanishing Riemannian curvature for the Levi-Civita connection and hence no gravitational field.

He now argues for a heuristic of finding field equations using the overdetermination approach. Since the tetrad field has  $n^2$  components, of which  $n$  remain undefined due to general covariance, one needs  $n^2 - n$  independent field equations. The heuristic principle of overdetermination is then stated like this:

On the other hand it is clear that a theory is all the more satisfying the more it restricts the possibilities (without getting into conflict with experience). The number  $Z$  of field equations hence shall be as large as possible. If  $\bar{Z}$  is the number of identities between them, then  $Z - \bar{Z}$  must be equal to  $n^2 - n$ .<sup>69</sup>

The identity that Einstein now put at the center of his derivation of field equations is related to identity (39) since it is similarly obtained using the commutation law (38) for covariant differentiation. But now he no longer used the quantity  $\mathcal{B}_{\kappa\lambda}^{\alpha}$  for the new divergence notation  $\dots/\alpha$  but rather looked directly at the commutation of the covariant derivatives for the torsion tensor  $\Lambda_{\alpha\nu}^{\mu}$ . This produced the identity

$$\Lambda_{\underline{\mu}\underline{\nu};\nu\alpha}^{\alpha} - \Lambda_{\underline{\mu}\underline{\nu};\alpha\nu}^{\alpha} - \Lambda_{\underline{\mu}\underline{\tau};\alpha}^{\sigma} \Lambda_{\sigma\tau}^{\alpha} \equiv 0, \quad (47)$$

where again the raising or lowering indices is indicated by underlining. Introducing the quantities

$$G^{\mu\alpha} \equiv \Lambda_{\underline{\mu}\underline{\nu};\nu}^{\alpha} - \Lambda_{\underline{\mu}\underline{\tau}}^{\sigma} \Lambda_{\sigma\tau}^{\alpha}, \quad (48)$$

$$F^{\mu\nu} \equiv \Lambda_{\underline{\mu}\underline{\nu};\alpha}^{\alpha} \quad (49)$$

<sup>68</sup> “Die Auffindung der einfachsten Feldgesetze, welchen eine Riemannsche Mannigfaltigkeit mit Fern-Parallelismus unterworfen werden kann” [Einstein, 1930a, p. 685].

<sup>69</sup> “Andererseits ist klar, daß eine Theorie desto befriedigender ist, je mehr sie die Möglichkeiten einschränkt (ohne mit Erfahrungen in Widerspruch zu treten). Die Zahl  $Z$  der Feldgleichungen soll also möglichst groß sein. Ist  $\bar{Z}$  die Zahl der zwischen diesen bestehenden Identitäten, so muß  $Z - \bar{Z}$  gleich  $n^2 - n$  sein” [Einstein, 1930a, p. 692].



the identity (47) can be rewritten as

$$G^{\mu\alpha}{}_{;\alpha} - F^{\mu\nu}{}_{;\nu} + \Lambda^{\sigma}{}_{\underline{\mu}\underline{\tau}} F^{\sigma\tau} = 0. \quad (50)$$

The field equations are now introduced as

$$G^{\mu\alpha} = 0, \quad (51)$$

$$F^{\mu\alpha} = 0. \quad (52)$$

As it stands the system of field equations does not satisfy Einstein's heuristic of overdetermination. Since  $F^{\mu\nu}$  is antisymmetric, Eqs. (51) and (52) represent  $n^2 + n(n-1)/2$  field equations that obey only  $n$  identities (47). In order to balance the number of equations and identities, Einstein proceeded to introduce an equivalent system of  $n^2 + n$  field equations. Rewriting identity (35) as

$$\Lambda^{\alpha}{}_{\kappa\lambda;\alpha} \equiv \phi_{\kappa,\lambda} - \phi_{\lambda,\kappa}, \quad (53)$$

he observed that (52) implies that  $\phi_{\alpha}$  may be obtained from a scalar potential  $\psi$ . Hence, (52) together with (53) is equivalent to the introduction of a field  $\Psi$ ,

$$\phi_{\alpha} = \frac{\partial \lg \psi}{\partial x^{\alpha}}, \quad (54)$$

which increases the number of variables to  $n^2 + 1$  but reduces the number of field equations to  $n^2 + n$ . Einstein still needed another identity which he derived by looking at the antisymmetric part  $\underline{G}^{\mu\alpha}$  of  $G^{\mu\alpha}$ . He obtained a set of  $n$  identities,

$$\left[ h\psi \left( 2\underline{G}^{\mu\alpha} - F^{\mu\alpha} + S^{\sigma}{}_{\underline{\mu}\underline{\alpha}} (\phi_{\sigma} - (\lg \psi)_{,\sigma}) \right) \right]_{\alpha} \equiv 0, \quad (55)$$

where  $S^{\sigma}{}_{\underline{\mu}\underline{\alpha}}$  is the completely antisymmetrized torsion (31). Of the  $n$  equations (55) only  $n-1$  are independent since the antisymmetry of  $[\dots]$  with respect to  $\alpha$  and  $\mu$  implies  $[\dots]_{,\alpha\mu} = 0$ , irrespective of any specific choice for  $G^{\mu\alpha}$  or  $F^{\mu\alpha}$ . Computing again the balance of the number of field equations ( $n^2 + n$ ) minus the number of (independent) identities ( $n + n - 1$ ) compared to the number of field variables ( $n^2 + 1$ ) minus the number of space-time dimensions to allow for general covariance  $n$ , these numbers now add up correctly as

$$(n^2 + n) - (n + n - 1) = (n^2 + 1) - n. \quad (56)$$

This essentially completed the derivation of the field equations (51), (52) as given in the *Annalen* paper. Actually Einstein was a bit more precise by arguing for the compatibility of the field equations on a hypersurface  $x^n = a$  and the possibility of a smooth continuation of all relations off the hypersurface.

A variational principle is no longer mentioned. In the final paragraph, Einstein looked at the first approximation of the field equations and derived relations that correspond to the Poisson equation and to the vacuum Maxwell equations, respectively.<sup>70</sup>

## 6. The final fate of the approach

The theory had now reached a stage where Einstein essentially stopped looking for other acceptable field equations, just as in the case of the prehistory of general relativity with publication of the *Entwurf*. And just as with the *Entwurf*, the *Annalen* paper represents the culmination of the distant parallelism approach. At this point, Einstein accepted the equations that he had come up with and proceeded to focus on their physical and mathematical consequences. This latter endeavor involved the elaboration of implications of physical significance such as the existence of particle-like solutions and their equations of motion. It also involved, again in perfect similarity with the *Entwurf*, the attempt to rederive the field equations from a variational principle and the investigation of their compatibility.

The final fate of the approach is documented by a French version of the *Annalen* article, three popular accounts of the present state of field theory that mention distant parallelism as a promising recent advance, and four further notes in the *Sitzungsberichte*, two of them co-authored with Walther Mayer.

### 6.1. Improving the derivation of the field equations

When the manuscripts for their *Annalen* papers were still sitting with the publisher, Cartan and Einstein had occasion for a personal encounter. In November 1929, Einstein travelled to Paris. He was awarded an honorary doctorate and also gave two lectures at the Institut Henri Poincaré.<sup>71</sup> Einstein's lectures at the Institut Henri Poincaré were subsequently published in French in the institute's *Annales* [Einstein, 1930b]. This French account of the theory closely parallels the version in the *Mathematische Annalen*, being slightly more explicit in the mathematical details.

The personal encounter between Einstein and Cartan also seemed to have resulted in some further work of the latter on the theory. This is evinced by a few extensive and technical manuscripts that have been published in the Einstein–Cartan correspondence [Debever, 1979]. One such manuscript [Debever, 1979, pp. 32–55] by Cartan immediately led Einstein to publish an improved version of the compatibility proof in his *Annalen* paper, even before that paper was available in print.<sup>72</sup> On December 12, 1929, Einstein

<sup>70</sup> This part of the *Annalen* paper is the subject of a later correspondence that took place in the late thirties between Einstein and Herbert E. Salzer, who wrote a master's thesis on “analytic, geometric and physical aspects of distant parallelism”. In this correspondence, Einstein admitted an error in the last section of his *Annalen* paper. But at that time, he had abandoned the approach long ago anyway. See [Salzer, 1974] for a detailed discussion.

<sup>71</sup> The lectures were given on 8 and 12 November, and the awarding of the honorary doctorates took place at the ceremony of the annual reopening of the Paris university at the Sorbonne on 9 November; see [Debever, 1979, pp. 21ff] for details.

<sup>72</sup> In a postscript to a letter to Cartan, dated 10 January, Einstein complains: “It is remarkable that the *Mathematische Annalen* has such terrible constipation that, after so many months, it has not been able to excrete what it has absorbed” [Debever, 1979, p. 121]. The correspondence between Einstein and Cartan at the end of 1929 was intense and it was Cartan who took the lead by working on the mathematical side of the problem. “I am very fortunate that I have acquired you as a coworker (Mit-Strebenden). For you have exactly that which I lack: an enviable facility in mathematics” [18 December 1929]. The correspondence with Cartan on teleparallelism recalls a similar correspondence between Einstein and Levi-Civita on mathematical details of the

submitted a communication to the Prussian Academy on the “Compatibility of the Field equations in the Unified Field Theory” [Einstein, 1930f]. In this short note, Einstein first gave a few critical remarks on his earlier papers. These concerned the divergence operation introduced in [Einstein, 1929b], which Einstein now considered inappropriate, because it did not vanish when applied to the fundamental tensor. Einstein also mentioned that the compatibility proof given in that paper was untenable because it erroneously assumed the existence of a set of identities for the field equations. Finally, Einstein pointed out that his discussion of the magnetic field equation in [Einstein, 1929e] was based on an unjustified assumption.

The major part of the note, however, was devoted to a brief survey of the mathematical apparatus of the theory (which Einstein probably gave because the long review paper had not yet come out) and a discussion of the compatibility issue. The main point was that Einstein had learned from Cartan that the compatibility proof could be improved.<sup>73</sup> The point was that the strange identity (55) could, in fact, be replaced by the simple identity

$$G^{\mu\alpha}_{;\mu} + \Lambda^{\alpha}_{\sigma\tau} G^{\sigma\tau} \equiv 0. \quad (57)$$

The compatibility proof was now given by Einstein for the field equations (51) and (52) on the basis of the identities (50), (53), and (57).

The issue of proving compatibility was taken up again in a very brief note from July 1930 [Einstein, 1930g], where Einstein again introduced a divergence operation  $\dots/\alpha$  and showed that it may be used to prove the compatibility of certain equations that are similar to his field equations. He did not, however, discuss the consequences for his system of equations (51) and (52) explicitly.

## 6.2. Promoting and defending the theory

In October 1929, Einstein was asked to substitute for the late secretary of state Leipart to give a lecture to some 800 invited members of the Kaiser-Wilhelm society and other representatives of scientific and cultural institutions and ministries. Einstein agreed and gave a talk on the *Problem of Space, Field, and Ether in Physics* on December 11, 1929.<sup>74</sup> Essentially the same talk was delivered to a large audience on the opening day of the *Second World Power Conference* which took place in Berlin from 16 to 25 June, 1930.<sup>75</sup> The text of this lecture was then published in the conference’s *Transactions* [Einstein, 1930d].<sup>76</sup> Just as in the articles of the New York and London *Times*, this lecture gave a historical account

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derivation of the *Entwurf* equations. Einstein seems to have had comparable feelings of appreciation for Levi-Civita to whom he wrote in 1917: “It must be nice to ride these fields on the cob of mathematics proper, while the likes of us must trudge along on foot.” (“Ich bewundere die Eleganz Ihrer Rechnungsweise. Es muss hübsch sein, auf dem Gaul der eigentlichen Mathematik durch diese Gefilde zu reiten, während unsereiner sich zu Fuss durchhelfen muss.”) [Einstein to Levi-Civita, 2 August 1917] [Einstein, 1998, Doc. 368].

<sup>73</sup> “Der Kompatibilitätsbeweis ist auf Grund einer brieflichen Mitteilung, welche ich Hrn. Cartan verdanke [...], gegenüber der in den Mathematischen Annalen gegebenen Darstellung etwas vereinfacht” [Einstein, 1930f, p. 18].

<sup>74</sup> [Harnack to Einstein, 18 October 1929, EA 1-084].

<sup>75</sup> [Körtgen to Einstein, 22 February 1930, EA 1-085].

<sup>76</sup> A similar popular account of *Space, Ether and the Field in Physics* was published in *Forum Philosophicum* [Einstein, 1930c] together with an English translation. Indeed, the text of the two penultimate paragraphs of this version and [Einstein, 1930d] that characterize the distant parallelism are identical. A two-page abbreviated version of [Einstein, 1930c] also mentions the distant parallelism approach [Einstein, 1930e].

of our concepts of space, starting with our prescientific notion, discussing Euclidean geometry, Cartesian analytic geometry, Newtonian absolute space, the ether concept of 19th-century electrodynamics, special relativity, and Riemannian geometry of general relativity. In the final paragraphs, Einstein hinted again at the latest progress of a “unitary field theory” based on a mathematical structure of space that is “a natural supplementation of the structure of space according to the Riemannian metric.” He explained again the meaning of distant parallelism and wrote, a little less self-confidently than in the *Times*,

For the mathematical expression of the field-laws we require the simplest mathematical conditions to which such a structure of space can conform. Such laws seem actually to have been discovered and they agree with the empirically known laws of gravitation and electricity in first approximation. Whether these field-laws will also yield a usable theory of material particles and of motions must be determined by deeper mathematical investigations. [Einstein, 1930c, p. 184]

Einstein also defended his new theory in private correspondence. A succinct example is a rebuttal of a saucy criticism by Wolfgang Pauli. With respect to the theory as presented in the *Annalen*, Pauli wrote that he no longer believed that the quantum theory might be an argument for distant parallelism after Weyl and Fock had shown that Dirac’s electron theory could be incorporated into a relativistic gravitation theory in a way that is not *globally* but *locally* Lorentz covariant. Pauli also wrote that he did not find the derivation of the field equations convincing, complained that the Maxwell equations would be obtained only in differentiated form, and expressed doubts as to whether an energy–momentum tensor of the field could be found. He finally missed the validity of the classical tests of general relativity, perihelion motion, and gravitational light bending. Pauli concluded

I would take any bet with you that you will have given up the whole distant parallelism at the latest within a year from now, just as you had given up previously the affine theory. And I do not want to rouse you to contradiction by continuing this letter, so as not to delay the approach of the natural decease of the distant parallelism theory.<sup>77</sup>

Einstein found Pauli’s critique “amusing but a little superficial.” Without going into details, he argued that Pauli was not in a position to “view the unity of the forces in nature from the correct standpoint” and one may not discard his theory before its mathematical consequences were thoroughly thought through. He claimed

that with a deeper look at it you would certainly understand that the system of equations advanced by me is forced by the underlying structure of space, particularly since the compatibility proof of the equations could be simplified in the meantime. Forget what you have said and engross yourself in the problem with such an attitude as though you had just come down from the moon and would yet need to form a fresh opinion. And then don’t utter an opinion before at least a quarter of a year has passed.<sup>78</sup>

<sup>77</sup> “[...] ich würde jede Wette mit Ihnen eingehen, dass Sie spätestens nach einem Jahr den ganzen Fernparallelismus aufgegeben haben werden, so wie Sie früher die Affintheorie aufgegeben haben. Und ich will Sie nicht durch Fortsetzung dieses Briefes noch weiter zum Widerspruch reizen, um das Herannahen dieses natürlichen Endes der Fernparallelismustheorie nicht zu verzögern” [Pauli to Einstein, 19 December 1929] [Pauli, 1979, Doc. 239].

<sup>78</sup> “Dass das von mir aufgestellte Gleichungssystem zu der zugrunde gelegten Raumstruktur in einer zwangsläufigen Beziehung steht, würden Sie bei tieferem Studium bestimmt einsehen, zumal der Kompatibilitätsbeweis der Gleichungen sich unterdessen

### 6.3. Elaboration of consequences

Both the long review paper in the *Annalen* (as well as its French counterpart [Einstein, 1930b]) and this short note end with the expression of the next step of the teleparallel approach:

The most important question that is now tied to the (rigorous) field equations is the question of the existence of singularity-free solutions which can represent electrons and protons.<sup>79</sup>

This problem was indeed attacked by Einstein in his pursuit of the teleparallel program. It was a problem where he found help from a collaborator. With Grommer and Müntz leaving for Minsk and Leningrad, Einstein may have found himself in need of new collaborators. After contacting Richard von Mises about suitable candidates, he was recommended Walther Mayer (1887–1948), then *Privatdozent* for mathematics in Vienna.<sup>80</sup> Mayer was an expert in invariant theory and differential geometry. Einstein was interested and technical arrangements were quickly agreed upon. Mayer arrived in Berlin some time in January 1930 but apparently began to work on problems associated with the teleparallelism approach before his arrival.<sup>81</sup>

The collaboration with Mayer proved to be of immediate success. Already on 20 February 1930, they presented a first joint paper for publication in the Academy proceedings [Einstein and Mayer, 1930]. In it they discussed two special solutions for the teleparallel field equations, i.e., those presented and derived in the *Annalen* paper [Einstein, 1930a], the case of spatially spherical symmetry, and the static case of an arbitrary number of nonmoving, noncharged mass points.

Assuming spatial rotation symmetry as well as reflection symmetry, their solution explicitly reads

$$\begin{aligned} h_s^\alpha &= \frac{\delta_s^\alpha}{\sqrt[4]{1 - \frac{e^2}{r^4}}}, \quad \alpha, s = 1, 2, 3, \quad h_s^4 = 0, \\ h_4^\alpha &= \frac{e}{\sqrt[4]{1 - \frac{e^2}{r^4}}} \frac{x^\alpha}{r^3}, \quad \alpha = 1, 2, 3, \quad h_4^4 = 1 + m \int \sqrt[4]{1 - \frac{e^2}{r^4}} \frac{dr}{r^2} \end{aligned} \quad (58)$$

where  $r^2 = \sum_{a=1}^3 x^a x_a$  is the spatial distance from the origin, and  $e$  and  $m$  two constants to be identified with the charge and mass of the particle.

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noch hat vereinfachen lassen. Vergessen Sie, was Sie gesagt haben, und vertiefen Sie sich einmal mit solcher Einstellung in das Problem, wie wenn Sie soeben vom Mond heruntergekommen wären und sich erst frisch eine Meinung bilden müssten. Und dann sagen Sie erst etwas darüber, wenn mindestens ein Vierteljahr vergangen ist" [Einstein to Pauli, 24 December 1929] [Pauli, 1979, Doc. 240].

<sup>79</sup> "Die wichtigste an die (strengen) Feldgleichungen sich knüpfende Frage ist die nach der Existenz singularitätsfreier Lösungen, welche die Elektronen und Protonen darstellen könnten" [Einstein, 1930f, p. 23].

<sup>80</sup> [Richard von Mises to Einstein, 17 December 1929, EA 18-225]. For biographical information on Mayer, see [Pais, 1982, pp. 492–494].

<sup>81</sup> See [Einstein to Mayer, 1 January 1930, EA 18-065].

For vanishing charge  $e$ , the solution reduces to

$$h_s^\alpha = \delta_s^\alpha, \quad s = 1, 2, 3, \quad h_4^\alpha = \delta_4^\alpha \left( 1 + \sum_j \frac{m_j}{r_j} \right), \quad m_j = \text{const.} \quad (59)$$

Einstein and Mayer interpreted (59) to the effect that two or more uncharged massive particles could stay at rest with arbitrary distance from each other. They emphasized, however, that the solution was singular and that the theory would not allow one to derive equations of motion for such singular solutions in contrast to the requirement that only nonsingular solutions be interpreted as elementary particles.

#### 6.4. *The demise of the Fernparallelismus approach*

Roughly a decade later, Einstein summarized his reasons for abandoning the distant parallelism approach:

Today, I am firmly convinced that the distant parallelism does not lead us to an acceptable representation of the physical field. From the reasons for this I will only give two:

- (1) One cannot find a tensor-like representation of the electromagnetic field.
- (2) The theory leaves too large a freedom for the choice of the field equations.<sup>82</sup>

The first point mentioned by Einstein is somewhat arcane to me. Other than the ambiguity of identifying the electromagnetic field, mentioned in passing in his second note (see the discussion above in Section 2.2), and the difficulty of establishing the equivalence to Maxwell's electrodynamics, as pointed out to him by Müntz and Lanczos (see the discussion above in Section 4.1), I did not find any more explicit discussion in his published papers or correspondence on this topic. Perhaps a more detailed examination of Einstein's unpublished research manuscripts (see footnote 2 above) will provide further insight on this point.

But the second point is, I believe, well illustrated by Einstein's last paper on this approach. It is again a paper coauthored with Mayer, and it is concerned with a "systematic investigation of compatible field equations that can be set in a Riemannian space with distant parallelism" [Einstein and Mayer, 1931a]. The paper is remarkable in two respects. For one, it was presented to the Academy on 23 April, 1931, and hence appeared some 9 months later than the last two-page note from July 1930. All other papers on the approach were published within at most 6 months in between. Even in the pure chronology, the paper thus appears as a belated and final word on the fate of the approach. Second, this paper, as we will see, is a quite unusual paper for Einstein in its technicality.

To discuss the admissible field equations, Einstein and Mayer demand—"as always" but without further justification—that these be linear in the second derivatives of the field variables  $h_{sv}$  and at most

<sup>82</sup> "Ich bin heute fest davon überzeugt, daß der Fern-Parallelismus zu keiner brauchbaren Darstellung des physikalischen Feldes führt. Von den Gründen will ich nur zwei anführen:

(1) Man gelangt nicht zu einer tensor-artigen Darstellung des elektromagnetischen Feldes.

(2) Die Theorie läßt eine zu große Freiheit für die Wahl der Feldgleichungen" [Einstein to Salzer, 29 August 1938] [Salzer, 1974, p. 90].



quadratic in the first derivatives. They also argue that the identities which the left sides of the field equations satisfy should contain these variables only linearly and in first order, and they also should contain the torsion tensor  $\Lambda_{\mu\nu}^\alpha$  explicitly only linearly. Using the notation of the previous papers, Einstein and Mayer now make the following *Ansatz* for the field equations of the theory,

$$0 = G^{\mu\alpha} = p\Lambda_{\underline{\mu}\underline{\nu};\underline{\nu}}^\alpha + q\Lambda_{\underline{\alpha}\underline{\nu};\underline{\nu}}^\mu + a_1\phi_{\underline{\mu};\underline{\alpha}} + a_2\phi_{\underline{\alpha};\underline{\mu}} + a_3g^{\mu\alpha}\phi_{\underline{\nu};\underline{\nu}} + R^{\mu\alpha}, \quad (60)$$

where  $p, q, a_1, a_2, a_3$  are arbitrary real coefficients, and  $R^{\mu\alpha}$  denotes an as yet unspecified term that is quadratic in the  $\Lambda$ 's.

They also write the divergence identity that is to be satisfied in the following general form

$$\begin{aligned} 0 \equiv G_{;\underline{\mu}}^{\mu\alpha} + AG_{;\underline{\mu}}^{\mu\alpha} + G^{\sigma\tau}(c_1\Lambda_{\sigma\tau}^\alpha + c_2\Lambda_{\sigma\underline{\alpha}}^\tau + c_3\Lambda_{\tau\underline{\alpha}}^\sigma) + c_4G^{\alpha\sigma}\phi_\sigma + c_5G^{\sigma\alpha}\phi_\sigma \\ + c_6G^{\sigma\sigma}\phi_{\underline{\alpha}} + BG_{;\underline{\alpha}}^{\sigma\sigma}, \end{aligned} \quad (61)$$

where again  $A, c_1, \dots, c_6$ , and  $B$  are unspecified coefficients.

Einstein and Mayer explicitly admit the possibility of other terms not contained in this *Ansatz*, especially for  $n = 4$  dimensions. Nevertheless they claim that the neglected terms would be “rather unnatural ones” (“wenig natürlich gebildete Glieder”) and that the general *Ansatz* of Eqs. (60), (61) is, in fact, the most general one that is consistent with the restrictive conditions of the problem.

Accepting the generality of the *Ansatz*, the problem of finding the manifold of admissible field equations then reduces to the algebraic problem of determining the 13 unspecified constants  $p, q, a_1, a_2, a_3, A, c_1, \dots, c_6$ , and  $B$ , as well as the constants implicitly contained in the generic term  $R^{\mu\alpha}$ . This algebraic problem is straightforward but formidable. One may well imagine that it took Einstein and Mayer a while to find their way through the resulting explicit equations.<sup>83</sup> The result is the subject of this final note on the *Fernparallelismus* approach. Introducing a few simplifications, they nevertheless end up with a system of 20 algebraic equations for 11 coefficients, which they list and discuss. Using a tree-like graphical representation, they classify possible types of solutions, which they subsequently try to associate with known cases and solutions.

The final upshot of their investigation is summarized in the final paragraph of their paper.

The result of the whole investigation is the following: In a space with Riemann-metric and *Fernparallelismus* of the character defined by (1), (2) [i.e., our Eqs. (60), (61)—TS] there are all in all four (nontrivial) different types of (compatible) field equations. Two of these are (nontrivial) generalizations of the original field equations of gravitation, one of which is already known as resulting from a Hamiltonian principle [cf. (10) and (11)]. The remaining two types are denoted in the paper by (13) and  $\Pi_{221221}$ .

These are Einstein's final words in print on the *Fernparallelismus* approach. Equations (10), (11), (13) of their paper that they refer to and the expression  $\Pi_{221221}$  indicate various field equations given in more or less explicit form.

<sup>83</sup> A number of apparently related but otherwise unidentified manuscript pages are extant in the Einstein Archives; see, e.g., EA 62-003ff, EA 62-054ff, EA 62-132ff.

## 7. Concluding remarks

As indicated in the introduction and at various points in the paper, the life cycle of the *Fernparallelismus* approach shows a number of similarities with the life cycle of the *Entwurf* theory of the years 1912–1915. For the sake of the present account, I would like to recall the following features of the fate of the *Entwurf* theory, the genesis, life, and demise of which is well understood by recent historical research.<sup>84</sup>

The theoretical framework of this theory crucially depended upon the insight that the metric tensor field is the mathematical ingredient needed to set up a generalized theory of relativity and a theory of gravitation. This insight was made by Einstein some time in the summer of 1912 but the mathematics associated with the metric tensor field was not fully understood by Einstein in the beginning. Somewhat fortuitously he was able to enter into an intense collaboration with a mathematician friend, Marcel Grossmann, then his colleague at the Polytechnic in Zurich. The subsequent development, as documented mainly by unpublished manuscripts and correspondence, consisted in an intense search for a gravitational field equation that would satisfy a number of heuristic requirements. An analysis of Einstein's research notes of that period showed that he pursued a dual strategy for finding field equations. At one point, Einstein was content with a set of field equations that was not generally covariant but seemed to square best with most of his other heuristic requirements. Einstein and Grossmann published their theory in their joint *Entwurf*. The further development of this theory involved both the elaboration of empirically relevant consequences, notably the planetary perihelion anomaly, and the further mathematical justification of its field equations, with particular emphasis on the question of their uniqueness. By mid-1915 several difficulties of the theory had become evident to Einstein, and it was then abandoned in November 1915 and superseded by new, generally covariant field equations, the Einstein equations of today's general relativity.

Reflecting on the “biographical” similarities between the *Entwurf* theory and the *Fernparallelismus* theory, it seems that there is a *systematic* reason for this similarity.<sup>85</sup> It resides in the roles that the mathematical representation in terms of the metric tensor field, respectively of the tetrad field, and the search for field equations for these quantities played in each theory.

In both cases, the mathematics associated with the new concept was poorly understood by Einstein in the beginning. In both cases, the mathematics had been worked out before in purely mathematical contexts. In both cases, it was through the mediation of more mathematically trained colleagues that Einstein learned about the earlier relevant mathematical developments. More specifically, we observe that after a relatively brief period where the mathematical concept of metric respectively tetrad was accepted as the key element, the further research program focussed on finding field equations for these quantities. In the attempts to find, derive, and justify those field equations, heuristic convictions become visible that had been conceived in previous work.

<sup>84</sup> For a historical account of the prehistory of general relativity along the lines given here, see [Renn and Sauer, 1999]. See also [Norton, 1984; Stachel, 2002, Chapter V; Renn et al., forthcoming] as well as further references cited in these works.

<sup>85</sup> I agree with the general thesis of [van Dongen, 2002a] who identified methodological convictions for Einstein's work on semivectors and on five-dimensional field theory that had originated during the *Entwurf* period. In contrast to van Dongen, I would only emphasize the constraints and inherent possibilities of the mathematical representation over the role of explicit methodological reflections.

In the case of the *Entwurf* theory, the relevant heuristic assumptions could be identified as the equivalence hypothesis and postulates of general covariance, energy–momentum conservation, and correspondence, i.e., the admissibility of the Newtonian limit [Renn and Sauer, 1999].

In the case of the *Fernparallelismus* approach, the corresponding heuristic convictions still need to be identified more precisely through the study of unpublished correspondence and notes. It appears, however, that one may similarly identify a number of postulates that play a similar role. Two such postulates are the demands of distant parallelism and general covariance. We also have a postulate that the known cases of the relativistic vacuum gravitational field equations and the Maxwell equations shall be identifiable in some weak field limit. Third, we have seen that Einstein postulated that nonsingular, spatially symmetric, stationary solutions can be found that can be interpreted as elementary particles. Finally, he was postulating that equations of motion should be derivable for those particle-like solutions.

In the case of the *Entwurf* theory, the heuristic postulates were mutually incompatible in Einstein's original understanding. The incompatibility showed itself in Einstein's difficulty in finding field equations that would satisfy all four of his postulates at the same time. As a consequence, Einstein developed a double strategy of finding field equations that we have called the mathematical, respectively the physical strategy [Renn and Sauer, 1999]. One strategy started from the postulates of general covariance and tried to modify equations constructed on the basis of the Riemann tensor in order to justify the more physically motivated postulates of energy–momentum conservation and of obtaining the Newtonian limit. The complementary strategy started from expressions that guaranteed the Newtonian limit from the beginning and tried to enlarge the covariance group to generalize the relativity principle.

In the *Fernparallelismus* approach something similar seems to be observable. Here again, we may distinguish two distinct approaches to the problem of finding field equations. A mathematical, variational approach started from a mathematically well-defined *Ansatz*, but the problem was to obtain the gravitational and electromagnetic field equations in first approximation. The complementary physical strategy, the overdetermination approach, on the other hand, started from identities that guaranteed the validity of the gravitational and electromagnetic equations from the outset. The drawback here was the mathematical problem of proving the compatibility of the field equations. In both cases, at the mature stage, Einstein settled for the more physical approach.

Can we also compare the demise of the two theories? From the more global perspective of Einstein's heuristics, the result of the final paper [Einstein and Mayer, 1931a] may be phrased as follows. The overdetermination approach to finding field equations within the distant parallelism framework had provided a manifold of different admissible equations. These were not only difficult to find and handle in their algebraic complexity. The approach also seemed to encompass the equations produced by the alternative variational approach and to produce even more admissible field equations than that method.

In the case of the *Entwurf* theory of gravitation, several difficulties accumulated before its demise. But what sealed the fate of the *Entwurf* in the end was the success of its alternative, the generally covariant Einstein equations [Renn and Sauer, 1999, pp. 115ff]. These equations gave the correct value for the anomaly of the perihelion motion for Mercury and they solved the energy–momentum problem by virtue of the contracted Bianchi identities.

More than one reason was presumably responsible for Einstein's loss of faith in the distant parallelism approach. The mere algebraic complexity can hardly have been the decisive reason for giving it up, certainly not from a logical point of view. But it may have motivated Einstein to explore alternatives. More problematic must have been the apparent impossibility to justify a set of field equations uniquely. But here again it is hard to see how this difficulty could be turned into a logically compelling argument for

giving up the approach. After all, one could always add new heuristic requirements, or justify particular equations *post hoc*, as it were, by their subsequent success. But just as in the case of the *Entwurf*, the final demise may have been effected by the success of a different theory.

Indeed, only a few months later Einstein and Mayer presented a new approach to a unified theory [Einstein and Mayer, 1931b] that may have seemed more promising to them at the time. In this approach, the introduction of an independent orthonormal basis field in some vector spaces associated with each point of the manifold is again the crucial mathematical ingredient. But now the frame fields and hence the vector spaces were no longer assumed to be of the same dimension as the underlying manifold and hence they were no longer to be identified with the tangent bundle. They were now taken to be five-dimensional. The introduction of a five-dimensional frame bundle pointed to a reconsideration of the Kaluza–Klein approach. Since the underlying space–time manifold was still assumed to be four-dimensional, the new approach was also sufficiently different from earlier consideration of the five-dimensional field theory that earlier arguments against the Kaluza–Klein approach were no longer valid. Indeed, the five-dimensional vector spaces may have seemed promising enough to justify the abandoning of the *Fernparallelismus* approach for the time being. In contrast to other approaches in his quest for a unified theory, it seemed to have been a final demise, too. Einstein apparently did not return to an exploration of the conceptual framework of distant parallelism in his subsequent quest for a unified field theory of gravitation and electromagnetism.

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## Appendix A. Note on notation

During the elaboration of the teleparallelism approach Einstein introduced—and dropped—a few notational idiosyncrasies. For a systematic reconstruction of the theory, these notational changes are awkward to deal with. However, for a historical reconstruction they provide very useful information. They help to identify and date calculational manuscripts and they may provide clues as to Einstein's reception of literature as well as to his heuristics.

I will summarize here the three notational peculiarities associated with the *Fernparallelismus* approach. They concern (a) the notation of the anholonomic indices of the tetrads, (b) a “new” divergence operation, and (c) a peculiar way of indicating raising and lowering of indices.

Einstein rather consistently denotes the anholonomic indices (*Bein-Indizes*) of the tetrads by Latin indices and the holonomic indices (*Koordinaten-Indizes*) by Greek indices. As discussed above in Section 3.2, Weitzenböck had written to Einstein shortly after the publication of Einstein's first two notes on teleparallelism, pointing out his priority with respect to the Weitzenböck connection. In Einstein's

next publications, in the Stodola-Festschrift [Einstein, 1929a] and in [Einstein, 1929b], he already used Weitzenböck’s notation of putting the anholonomic index to the left of the tetrad symbol,  ${}^sh_\mu$ , with explicit reference to Weitzenböck’s paper. The notation is used again, but for the last time, in March 1929 in [Einstein, 1929e]. The *Annalen* paper of summer 1929 reverts to the previous right-hand-side notation. The left-hand-side notation therefore should give a fairly accurate hint to material dating between summer 1928 and summer 1929.

In his note from January 1929 [Einstein, 1929b], Einstein introduced what he called a “divergence” of a tensor density  $\mathfrak{A} \equiv hA$ ,  $h \equiv \det(h_{s\mu})$  by the following definition:

$$\mathfrak{A}^{\sigma\cdots i}_{\tau\cdots/i} = \mathfrak{A}^{\sigma\cdots i}_{\tau\cdots,i} + \mathfrak{A}^{\alpha\cdots i}_{\tau\cdots} \Delta^\sigma_{\alpha i} + \cdots - \mathfrak{A}^{\sigma\cdots i}_{\alpha\cdots} \Delta^\alpha_{\tau i} - \cdots. \quad (\text{A.1})$$

Here a subscript comma denotes ordinary coordinate differentiation and the dots indicate further contravariant and covariant indices. The new “divergence” coincides with the usual covariant divergence  $\mathfrak{A}^{\sigma\cdots}_{\tau\cdots;\sigma}$  formed using the covariant derivative associated with the Weitzenböck connection  $\Delta$ , see (6), for the case of vanishing torsion:

$$\mathfrak{A}^{\sigma\cdots}_{\tau\cdots;\sigma} \equiv \mathfrak{A}^{\sigma\cdots}_{\tau\cdots/\sigma} + \mathfrak{A}^{\sigma\cdots}_{\alpha\cdots} \Delta^\sigma_{\alpha\sigma}. \quad (\text{A.2})$$

Heuristically, it was introduced in the context of introducing the overdetermination approach because the relevant identities take on a compact form using this notation. Einstein used this notation again in his note from March 1929 in which he goes back to the Hamilton approach. However, in the *Annalen* paper, he explicitly wrote that he no longer recognized a specific physical meaning of that divergence operation.<sup>86</sup>

Strangely enough, Einstein did revert to this nonstandard divergence another time. In his short, two-page note [Einstein, 1930g] he reintroduced the divergence symbol for an arbitrary tensor  $A^\nu$ ,

$$A^{\nu}_{/\nu} = A^{\nu}_{;\nu} - A^\nu \varphi_\nu, \quad (\text{A.3})$$

where  $\varphi_\nu \equiv \Delta^\alpha_{\alpha\nu}$ . It is also used, albeit rather inconspicuously, in two equations in [Einstein and Mayer, 1931a].

The third notational idiosyncrasy was also introduced in the January 1929 note and was used in all subsequent papers on *Fernparallelismus*.

Sometimes I will indicate the raising respectively lowering of an index by underlining the corresponding index.<sup>87</sup>

An explicit example is (cf. Einstein, 1930a, p. 693)

$$\begin{aligned} \Lambda^\alpha_{\underline{\mu}\underline{\nu}} &\equiv \Lambda^\alpha_{\beta\gamma} g^{\mu\beta} g^{\nu\gamma}, \\ \Lambda^\alpha_{\underline{\mu}\underline{\nu}} &\equiv \Lambda^\beta_{\mu\nu} g_{\alpha\beta}. \end{aligned} \quad (\text{A.4})$$

<sup>86</sup> “In früheren Arbeiten habe ich noch andere Divergenzoperatoren eingeführt, bin aber davon abgekommen, jenen Operatoren eine besondere Bedeutung zuzuschreiben” [Einstein, 1930a, p. 689].

<sup>87</sup> “Ich will manchmal das Heraufziehen bzw. Hinunterziehen eines Index dadurch andeuten, daß ich den betreffenden Index unterstreiche” [Einstein, 1929b, p. 3].

## Appendix B. Modern characterization of teleparallelism as a geometric structure

The basic ingredients are a bare differentiable manifold, a curvature-free connection that permits definition of a frame field on the tangent bundle, and the demand of global  $SO(n-1, 1)$ -symmetry that allows one to define a metric tensor field in a meaningful way. Naturally, a coordinate-free characterization raises issues of global existence and similar concerns which, however, I will not discuss here.

**Step 0.** The starting point is an  $n$ -dimensional real, differentiable,  $C^\infty$ -manifold  $M$  just as in any modern account of the mathematical structure of general relativity; i.e., if needed, one might specify it as paracompact, Hausdorff, etc.

**Step 1.** Let  $\vartheta_a$  be a frame field on  $M$ , i.e., a set of  $n$  linearly independent, differentiable vector fields or, in other words, a cross section of the frame bundle. Such a frame field may not exist globally. If that is the case, we restrict ourselves to a parallelizable subset of  $M$ . At this point,  $\vartheta_a$  is not specified. It will be obtained later as a solution to some set of field equations.

We now introduce a connection, i.e., a  $\mathfrak{gl}(\mathbb{R}, n)$ -valued one-form  $\omega_b^a$  on the tangent bundle  $TM$  that is compatible with  $\vartheta_a$ , in the sense that the associated parallel transport is realized by the field  $\vartheta_a$ ; i.e., the covariant derivative of the frame vectors vanishes.

This condition determines the connection uniquely. By patching together information from different coordinate charts, it can also be defined globally, even if a global frame field does not exist. Without historical prejudice (see Section 3.2), we shall call the connection the Weitzenböck connection. The curvature form  $O_b^a = d\omega_b^a + \omega_m^a \wedge \omega_b^m$  for this connection vanishes,  $O_b^a \equiv 0$ . Its torsion two-form  $\Omega_a = d\vartheta_a - \vartheta_m \wedge \omega_a^m$ , however, does not vanish in general. Conversely, a Weitzenböck connection  $\omega_b^a$  does not uniquely determine a frame field  $\vartheta_a$ . The frame field is only determined up to a global  $GL(\mathbb{R}, n)$ -transformation.

The global  $GL(\mathbb{R}, n)$ -symmetry can also be seen like this. Given a Weitzenböck connection over  $M$  and a local frame at a single point  $p \in M$ , we can parallel transport the frame over the tangent bundle and construct a frame field  $\vartheta_a$ . Obviously, we could start with any linearly independent set of vectors in  $T_p M$  and would obtain different frame fields for each such frame in  $T_p M$ , which are equivalent up to global Lorentz rotations.

**Step 2.** Given the frame field, we can now define a metric by conceiving of the frame field as an orthonormal vector field. To be specific, we will now assume the manifold to be of dimension  $n = 4$ . We then obtain a metric tensor field by

$$g = o^{ab} \vartheta_a \vartheta_b, \quad (\text{B.1})$$

where

$$o^{ab} = \text{diag}(-1, -1, -1, +1). \quad (\text{B.2})$$

The definition of the metric reduces the global  $GL(\mathbb{R}, 4)$ -symmetry to a global  $SO(3, 1)$ -symmetry. This symmetry requirement defines Einstein's version of distant parallelism. Any frame field  $\vartheta_a$  uniquely defines a metric tensor field. The converse is not true, since a metric tensor field is determined by  $n(n+1)/2$  components, whereas a set of  $n$  linear independent vectors is defined by  $n^2$  components.



In  $n = 4$  dimensions, the metric tensor determines 10 of the 16 components leaving 6 components undetermined. This is just the amount of freedom needed to accommodate the electromagnetic field in the theory.

The existence of a metric tensor field on  $M$  allows the definition of a second, uniquely defined, metric compatible connection, i.e., the usual Levi-Civita connection. Its torsion two-form vanishes while its curvature two-form in general does not vanish. The curvature associated with the Levi-Civita connection vanishes if and only if the Weitzenböck torsion vanishes.<sup>88</sup>

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<sup>88</sup> Abraham Pais observed that the distant parallelism approach was unusual for Einstein because “the most essential feature of the ‘old’ theory is lost from the very outset: the existence of a nonvanishing curvature tensor” [Pais, 1982, p. 344]. Clearly, this remark is in need of qualification, since it ignores the fact that more than one connection plays a role in this theory, and that the Levi-Civita connection does imply, in general, nonvanishing curvature.



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